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Finitary Spacetime Sheaves of Quantum Causal Sets: Curving Quantum Causality

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A locally finite, causal, and quantal substitute for a locally Minkowskian principal fiber bundle $\mathcal{P}$ of modules of Cartan differential forms Ω over a bounded region X of a curved C^{∞}-smooth spacetime manifold M with structure group $\mathbf{G}$ that of orthochronous Lorentz transformations $L^{+} := SO(1, 3)^{\uparrow}$, is presented. $\mathcal{P}$ is usually regarded as the kinematical structure of classical Lorentzian gravity when the latter is viewed as a Yang-Mills type of gauge theory of a $sl(2, \mathbb{C})$-valued connection 1-form $\mathcal{A}$. The mathematical structure employed to model this replacement of $\mathcal{P}$ is a principal finitary spacetime sheaf $\vec{\mathcal{P}}_n$ of quantum causal sets $\vec{\Omega}_n$ with structure group $\mathbf{G}_n$, which is a finitary version of the continuous group $\mathbf{G}$ of local symmetries of General Relativity, and a finitary Lie algebra $\mathbf{g}_n$-valued connection 1-form $\mathcal{A}_n$ on it, which is a section of its subsheaf $\vec{\Omega}^1_n$. $\mathcal{A}_n$ is physically interpreted as the dynamical field of a locally finite quantum causality, whereas its associated curvature $\mathcal{F}_n$ as some sort of "finitary and causal Lorentzian quantum gravity."

> . . . The locality principle seems to catch something fundamental about nature . . . Having learned that the world need not be Euclidean in the large, the next tenable position is that it must at least be Euclidean in the small, a manifold. The idea of infinitesimal locality presupposes that the world is a manifold. But the infinities of the manifold (the number of events per unit volume, for example) give rise to the terrible infinities of classical field theory and to the weaker but still pestilential ones of quantum field theory. The manifold postulate freezes local topological degrees of freedom which are numerous enough to account for all the degrees of freedom we actually observe.
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> *The next bridgehead is a dynamical topology, in which even the local topological structure is not constant but variable.*[4] The problem of enumerating all topologies of infinitely many points is so absurdly unmanageable and unphysical that dynamical

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[4] Our emphasis in order to prepare the reader for what will follow in the present paper.

topology virtually forces us to a more atomistic conception of causality and space-time than the continuous manifold . . . (Finkelstein, 1991).

1. INTRODUCTION CUM PHYSICAL MOTIVATION

We are still in need of a cogent quantum theory of gravity. A quantum field-theoretic scenario for General Relativity (GR) is assailed by nonrenormalizable infinities coming from the singular values of fields that are assumed to propagate and interact on a smooth spacetime manifold. Most likely, it is our modeling of spacetime after a C^∞-smooth differential manifold that is the culprit for this unpleasant situation. We can hardly expect Nature to have any infinities, but we can be almost certain that it is our own theoretical models of Her that are of limited applicability and validity.

The present paper takes a first step towards arriving at an operationally sound, locally finite, causal, and quantal model of classical Lorentzian gravity from a finitary spacetime sheaf (finsheaf)-theoretic point of view. Classical Lorentzian gravity may be regarded as a Yang-Mills type of gauge theory of a $sl(2, \mathbb{C})$-valued connection 1-form $\mathcal{A}$ and its kinematics is suitably formulated in a locally Minkowskian principal fiber bundle $\mathcal{P}$ of modules of Cartan differential forms Ω over a bounded region X of a curved C^∞-smooth differential manifold spacetime M with continuous structure group $\mathbf{G}$ that of orthochronous Lorentz transformations $L^+ := SO(1, 3)^\uparrow$. A principal finsheaf $\vec{\mathcal{P}}_n$ of quantum causal sets (qausets[5]) $\vec{\Omega}_n$ having as structure group a finitary version $\mathbf{G}_n$ of L^+, together with a finitary spin-Lorentzian connection $\mathcal{A}_n$ which is a $\mathbf{g}_n$-valued section of the subsheaf $\vec{\Omega}^1_n$ of reticular 1-forms of $\vec{\mathcal{P}}_n$, is suggested as a locally finite model, of strong algebraic-operational character, of the dynamics of the quantum causal relations between events and their local causal symmetries in a bounded region X of a curved smooth spacetime manifold M. In short, we propose $(\vec{\mathcal{P}}_n, \mathcal{A}_n)$ as a finitary, causal, and quantal replacement of the classical gravitational spacetime structure $(\mathcal{P}, \mathcal{A})$.[6] The theoretical model $(\vec{\mathcal{P}}_n, \mathcal{A}_n)$ is supposed to be a preliminary, because kinematical, step in yet another attempt at viewing the problem of quantum gravity as the dynamics of a local, finitistic, and quantal version of a variable causality or "causal topology"[7] (Bombelli *et al.*, 1987; Finkelstein, 1988, 1989, 1991, 1996; Raptis, in press, 2001, in preparation; Raptis and Zapatrin, in press; Sorkin, 1990a,b, 1995).

[5] Since "causal sets" are coined "causets" for short by Sorkin (private communication), "quantum causal sets" may be similarly nicknamed "qausets."

[6] Our scheme may be coined a "finitary and causal Lorentzian quantum gravity," although it is perhaps more precise to think of $\vec{\mathcal{P}}_n$ as a finitary, causal, and quantal substitute for the kinematical structure $\mathcal{P}$ on which GR is cast as a gauge theory, rather than directly of GR on it *per se*. For instance, we will go as far as to define curvature $\mathcal{F}_n$ on $\vec{\mathcal{P}}_n$, but we will not give an explicit expression of the dynamical Einstein equations on it. The latter is postoponed to another paper (Raptis, in preparation-c).

[7] See opening quotation above.

In more detail, the continuous (i.e., C^0) topology of a bounded region X of a spacetime manifold M has been successfully approximated by so-called "finitary topological spaces" which are mathematically modeled after partially ordered sets (posets) (Sorkin, 1991). The success of such coarse approximations of the topological spacetime continuum rests on the fact that an inverse system consisting of finer-and-finer finitary posets possesses, at the maximum (finest) resolution of X into its point events, a limit space that is effectively homeomorphic to X (Sorkin, 1991).

Similarly, coarse approximations of the continuous (i.e., C^0) spacetime observables on X have been soundly modeled after so-called "finitary spacetime sheaves"—"finsheaves" for short—which are structures consisting of continuous functions on X that are locally homeomorphic to the finitary posets of Sorkin (Raptis, 2000a). Here too, an inverse system of such finsheaves was seen to "converge," again at maximum refinement and localization of X into its point events, to $S(X)$—the sheaf of C^0-spacetime observables on X which is generated by (the germs of) its continuous sections (Raptis, 2000a).

In (Raptis and Zapatrin, 2000), an algebraic quantization procedure of Sorkin's finitary poset substitutes for continuous spacetime topology was presented, first by associating with every such poset P a noncommutative Rota incidence algebra $\Omega(P)$, then by quantally interpreting the latter's structure. The aforementioned limit of a net of such quantal incidence algebras was interpreted as Bohr's Correspondence Principle in the sense that the continuous spacetime manifold topology emerges, as a classical structure, from some sort of decoherence of the underlying discrete and coherently superposing quantum Rota-algebraic topological substrata (Raptis and Zapatrin, 2000, in press). The operationally pragmatic significance of the latter, in contradistinction to the ideal and, because of it, pathological[8] event structure that the classical topological manifold model of spacetime stands for, was also emphasized by Raptis and Zapatrin.

Furthermore, it has been argued (Raptis and Zapatrin, 2000, in press) that, in view of the fact that the $\Omega(P)$s were seen to be discrete differential manifolds in the sense of Dimakis *et al*. (1995), not only the continuous C^0-topological, but also the smooth (i.e., C^∞) differential structure of classical spacetime, emerges at the operationally ideal classical limit of finest resolution of a net of quantal incidence algebras. Since only at this ideal classical limit of an inverse system of such reticular quantum topological substrata the local structure of the differential spacetime manifold emerges,[9] the substrata were conceived as being essentially

[8] Due to the unphysical infinities in the form of singularities from which the classical and quantum field theories, which are defined on the operationally ideal and experimentally unrealistic spacetime continuum, suffer (see opening quotation).

[9] That is to say, the spacetime point event and the space of covariant directions tangent to it (i.e., its cotangent space of differential forms).

alocal structures (Raptis and Zapatrin, 2000), with this "a-locality" signifying some sort of independence of these algebraic structures from the classical conception of spacetime as a smooth background geometric base space. In a similar way, the finsheaf-theoretic approach developed in (Raptis, 2000a), with its finitary algebraic-operational character, strongly emphasizes the physical significance of such a noncommitment to an inert background geometrical base spacetime manifold, as well as its accordance with the general operational, ultimately "pragmatic" (Finkelstein, 1996), philosophy of Quantum Theory (QT).

Moreover, at the end of (Raptis, 2000a), it was explicitly mentioned that by assuming further algebraic structure for the stalks of the aforementioned finsheaves, as for instance by considering sheaves of incidence algebras over Sorkin's finitary topological posets, at the limit of maximum resolution or "fine-graining" of a net of such finsheaves of Rota algebras, which can also be regarded as Bohr's classical limit *à la* Raptis and Zapatrin (2000), the differential triad $(X, \Omega := \oplus_i \Omega^i, d)$ should emerge. The latter stands for the sheaf of modules of Cartan differential forms Ω on the smooth X, equipped with the nilpotent Kähler-Cartan differential operator d which effects (sub)sheaf morphisms of the following sort d: $(X, \Omega^i) \rightarrow (X, \Omega^{i+1})$ (Mallios, 1998a). Thus, a finsheaf of Rota incidence algebras is expected to be a sound model of locally finite, as well as quantal, "approximations" of the smooth spacetime observables—the classical spacetime dynamical fields.[10]

Parenthetically, and with an eye towards the physical interpretation to be given subsequently to our mathematical model, we should mention that the inverted commas over the word "approximations" in the last sentence may be explained as follows: after the successful algebraic quantization of Sorkin's discretized spacetimes in (Raptis and Zapatrin, 2000), it has become clear that the resulting alocal quantum topological incidence algebras $\Omega(P)$ associated with the finitary topological posets P in (Sorkin, 1991) should not be thought of as approximations proper of the classical smooth differential forms like their corresponding Ps or the finsheaves S_n in (Raptis, 2000a) actually approximate the C^0-topological manifold structure of classical spacetime, as if a geometric spacetime exists as a background base space "out there." Rather, they should be regarded as operationally pragmatic and relatively autonomous quantum spacetime structures an inverse system of which possesses an operationally ideal (i.e., unobservable in actual experiments) and classical, in the sense of Bohr, limit structure isomorphic to the differential manifold

[10] We tacitly assume that the classical model for the kinematics of spacetime and the fields inhabiting, dynamically propagating and interacting on it is that of a 4-dimensional differential (or C^∞-smooth) manifold M, with fiber spaces Ω^n of smooth Cartan exterior n-forms attached at or soldered on its point events. Physical fields are then modeled after cross-sections of this Cartan fiber bundle $\mathcal{P}$ of smooth exterior forms (Baez and Muniain, 1994; Göckeler and Schücker, 1990; Von Westenholz, 1981).

model of spacetime (Raptis and Zapatrin, 2000, in press). From this viewpoint, the quantum topological incidence algebras $\Omega(P)$ (and their qauset relatives in (Raptis, 2000b)) are regarded as being physically fundamental (primary) and their correspondence limit geometric point set manifold structure as being derivative (secondary), ultimately, their emergent classical correspondent or infinite localization energy limit space (Cole, 1972). Properly conceived, it is the classical theory (model) that should be thought of as an approximation of the deeper quantum theory (model), not the other way around (Finkelstein, 1996). Thus, "quantum replacements" or "quantum substitutes" instead of "approximations" will be used more often from now on to describe our finsheaves (of qausets), although it is fair to say that such combinatorial structures were initially conceived as approximations proper of the C^0-spacetime topology in (Raptis, 2000a), as it was originally motivated by Sorkin (1991).

In toto, this nonacceptance of ours of spacetime as a dynamically inactive smooth geometric receptacle of the physical fields or as a background stage that supports their dynamical propagation, that is passively existing as a static state space "out there" and whose structure is fixed *a priori* and forever by the theorist independently of our experimental actions on or operations of observation of "it," is the essence of the operationally sound quantum physical semantics that we will give to our algebraic finsheaf model in the present paper.

In GR—the classical theory of gravity which is based on the kinematical-structural assumption that spacetime is a 4-dimensional pseudo-Riemannian manifold M—the main dynamical variable is the smooth Lorentzian spacetime metric $g_{\mu\nu}$ which is physically interpreted as the gravitational potential. The local relativity group of GR, in its original formulation in terms of the Lorentzian metric $g_{\mu\nu}$, is the orthochronous Lorentz group $L^+ := SO(1, 3)^\uparrow$. GR may also be formulated in terms of differential forms on the locally Minkowskian bundle $\mathcal{P}$ (Göckeler and Schücker, 1990).[11] Equivalently, in its gauge-theoretic spinorial formulation (Baez and Muniain, 1994[12]; Bergmann, 1957[13]) gravity may be conceived as a type of gauge theory of a $sl(2, \mathbb{C})$-valued 1-form $\mathcal{A}$—the spin-Lorentzian connection field, which represents the gravitational gauge potential. A sound model for the

[11] See chapter on the Einstein-Cartan theory. We call $\mathcal{P}$ "the Cartan principal fiber bundle with structure group the orthochronous Lorentz group L^+ of local invariances of GR."

[12] We refer to Ashtekar's modfication of the Palatini formulation of GR by using new spin variables (Ashtekar, 1986). In this theory, only the self-dual part $\mathcal{A}^+$ of a spin-Lorentzian connection $\mathcal{A}$ is regarded as being physically significant. In (Raptis, in press) this is used as an example to argue that the fundamental quantum time asymmetry expected of "*the true quantum gravity*" (Penrose, 1987) is already built into the kinematical structure of a locally finite, causal, and quantal version of that theory modeled after curved finsheaves or schemes (Hartshorne, 1983; Shafarevich, 1994) of qausets.

[13] In this theory, $g_{\mu\nu}$ is replaced by a field of four 2×2 Pauli spin-matrices.

kinematics of this theory is a principal fiber bundle $\mathcal{P}$ over (the region X of) the C^∞-smooth spacetime manifold M, with structure group $\mathbf{G} = SL(2, \mathbb{C})$[14] and a nonflat connection 1-form $\mathcal{A}$ taking values in the Lie algebra $\mathbf{g} = sl(2, \mathbb{C})$ of $\mathbf{G}$[15], totally, $(X, \mathcal{P}, \mathbf{G}, \mathcal{A})$.[16] Thus, by the discussion in the penultimate paragraph, it follows that a principal finsheaf of quantum incidence algebras, together with a nonflat connection taking values in their local symmetries, may be employed to model a locally finite and quantal version of Lorentzian gravity in its gauge-theoretic formulation on a smooth spacetime manifold.

However, there seem to be *a priori* two serious problems with such a model. On the one hand, only Riemannian (i.e., positive definite) metric connections may be "naturally" defined on discrete differential manifolds such as our Rota incidence algebras (Dimakis and Müller-Hoissen, 1999), and on the other, the anticipated classical limit sheaf or fiber bundle (X, Ω, d) is flat (Mallios, 1998a).[17] The first comes into conflict with the indefinite character of the local spacetime metric of GR,[18] thus also with its local relativity group[19]; whereas the second, with the general relativistic conception of the gravitational field strength as the nonvanishing curvature of spacetime.

One should not be discouraged, for there seems to be a way out of this double impasse which essentially motivated us to consider finsheaves of qausets in the first place. First, to deal with the "signature problem," we must change physical

[14] A principal fiber bundle with structure group $\mathbf{G}$ may also be called a "$\mathbf{G}$-bundle" for short.

[15] Since locally in the group fiber (i.e., Lie algebra-wise in the fiber space) of the $\mathbf{G}$-bundle $\mathcal{P}$ $sl(2, \mathbb{C})$ is isomorphic to the Lie algebra $\ell^+ = so(1, 3)^\uparrow$ of the orthochronous Lorentz group L^+, $\mathcal{P}$ may equivalently be thought of as having the latter as structure group $\mathbf{G}$. Due to this local isomorphism $\mathcal{A}$ is given the epithet "spin-Lorentzian" and the same symbol $\mathcal{P}$ is used above for both the Cartan (Lorentzian) and the Bergmann (spin) $\mathbf{G}$-bundles. Thus, $\mathcal{P}$ is called "the CartanBergmann $\mathbf{G}$-bundle."

[16] The name "principal" is usually reserved only for the group $\mathbf{G}$-bundle or sheaf, whereas the vector or algebra sheaf that carries it, in our case Ω, is called "associated" (Mallios, 1998a). Here we use one symbol, $\mathcal{P}$, and one name, "principal," for both the $\mathbf{G}$-sheaf of orthochronous Lorentz transformations L^+ and its associated locally Minkowskian sheaf of differential forms Ω. Conversely, in section 4 we first define Ω as an algebra sheaf and then we coin the $\mathbf{G}$-sheaf of its Lorentz symmetries "adjoint." There should be no misunderstanding: Ω is associated with $\mathbf{G}$, or *vice versa*, $\mathbf{G}$ is adjoint to Ω, and together they constitute the principal sheaf $\mathcal{P}$. Nevertheless, we apologize to the mathematical purist for this slight change in nomenclature.

[17] Dimakis and Müller-Hoissen (1999) also mention the fact that the (torsionless) Riemannian metric connection ∇ of the universal differential calculus on a discrete differential manifold is flat in that it reduces to the nilpotent Kähler-Cartan differential d whose curvature $\mathcal{R}$ is zero, since $\mathcal{R} := \nabla^2 = d^2 = 0$.

[18] In GR, the local metric field $g_{\mu\nu}$ is Lorentzian (of signature 2), not Euclidean (of trace 4).

[19] The group of local isometries of GR, at least in its spinorial gauge-theoretic formulation mentioned above, is taken to be $SL(2, \mathbb{C})$—the double cover of the orthochronous Lorentz group $L^+ = SO(1, 3)^\uparrow$ of local invariances of $g_{\mu\nu}$ that also locally preserve the orientation of time, not the 4-dimensional unimodular Euclidean rotations in $SO(4)$. In this sense GR is a theory of (locally) Lorentzian gravity.

interpretation for the algebraic structure of the stalks of the aforementioned finsheaf of quantal incidence algebras from "topological" to "causal." This means that we should consider finsheaves of the qausets in (Raptis, 2000b), rather than finsheaves of the quantum discretized spacetime topologies in (Raptis and Zapatrin, 2000). Indeed, Sorkin (1995), in the context of constructing a plausible theoretical model for quantum gravity, convincingly argues for a physical interpretation of finitary posets as locally finite causets (Bombelli *et al.*, 1987; Sorkin, 1990a,b) and against their interpretation as finite topological spaces or simplicial complexes (Alexandrov, 1956a,b, 1961; Raptis and Zapatrin, 2000). Similar arguments against a nonrelativistic, spatial conception of topology and for a temporal or causal one which is also algebraically modeled with a quantum interpretation given to this algebraic structure, like the quantum causal topology of the qausets in (Raptis, 2000b), are presented in (Finkelstein, 1988). Ancestors of the causet idea are the classic works of Robb (1914), Alexandrov (1956a,b, 1967), and Zeeman (1964, 1967) which show that the topology and conformal geometry of Minkowski space $\mathcal{M}$, as well as its relativity group L^+ of global orthochronous Lorentz transformations modulo spacetime volume-preserving maps, can be determined by modeling the causal relation between its events after a partial order. Similarly in spirit, the derivation of the entire geometry of the Lorentzian spacetime manifold—the kinematical structure of GR—(i.e., its topology, dimensionality, differential and indefinite-Lorentzian metric structure) lies at the heart of the causet approach to quantum gravity propounded in (Bombelli *et al.*, 1987; Bombelli and Meyer, 1989; Sorkin, 1990a,b, 1995).

On the other hand, causality as a partial order, although it solves the "signature problem," is unable to adequately address the second "curvature problem" mentioned above, since it determines, up to a conformal (i.e., volume) factor, the Minkowski space $\mathcal{M}$ of Special Relativity (SR) and its Lorentz symmetries, which is flat and its Lorentz isometries are global. Our way out of this second "curvature impasse" involves a rather straightforward localization or gauging of the qausets in (Raptis, 2000b), by considering a nonflat connection $\mathcal{A}_n$ on a finsheaf of such quantally and causally interpreted incidence algebras, thus by emulating the work of Mallios (1998a,b, in press, in preparation)[20] that studies Yang-Mills guage connections on **G**-sheaves of vector spaces and algebras in general. This gauging of quantum causality translates in a finitary and quantal setting the fact that the classical theory of gravity, GR, may be regarded as SR localized or being gauged.[21] This connection variable is supposed to represent the dynamics of an atomistic local quantum causality as the latter is algebraically encoded stalk-wise in the finsheaf (i.e., in the qausets that dwell in these stalks). The result may be regarded

[20] Albeit, in a finitary causal and quantal context.

[21] So that the spacetime metric, or its associated (i.e., metric) connection, become dynamical field variables (Torretti, 1981) and are not fixed "trivial" constant entities throughout spacetime.

as the first essential step towards formulating a finitary dynamical scenario for the qauset stalks of the sheaf which, in turn, may be physically interpreted as a finitary and causal model of the still incompletely or not even well formulated theory of Lorentzian quantum gravity.

Equivalently, and in view of the sound operational interpretation given to the topological incidence algebras in (Raptis and Zapatrin, 2000) as well as to the topological finsheaves in (Raptis, 2000a), our model may be physically interpreted as locally finite and quantal replacements of the dynamics of the local causal relations between events and their local causal symmetries[22] in a limited (i.e., finite or bounded), by our own domain of experimental activity (i.e., laboratory) (Raptis and Zapatrin, 2000), region X of the smooth spacetime manifold M. As we mentioned above, the latter "exists" only as a "surrogate background space" that helps one remember where the discreteness of our model comes from, but it is not essential to the physical problem in focus. Again, the spacetime continuum, as a "base space," is only a geometrical scaffolding that supports our structures,[23] but that should also be discarded after their essentially alocal-algebraic, quantal-operational and causal (i.e., nonspatial, but temporal) nature is explicated and used for our problem in focus. Then, the aforementioned correspondence principle for quantal topological incidence algebras may be used on (an inverse system of) the principal finsheaves of qausets and their nonflat spin-Lorentzian connections in order to recover the classical spacetime structure on which GR is formulated, as the classical theory of gravity, at the classical and operationally ideal limit of infinite energy of resolution (Cole, 1972) (i.e., of infinite power of localization) of spacetime into its point events. This classical kinematical limit spacetime model for GR, as a gauge theory, is the one mentioned above, namely, a principal fiber bundle $\mathcal{P}$ of modules of smooth Cartan differential forms Ω, over (a region X of) a C^∞-smooth Lorentzian spacetime manifold M, with structure group $\mathbf{G} = SL(2,\mathbb{C})$ or its locally isomorphic $SO(1,3)^{\uparrow}$, and a nonflat $sl(2,\mathbb{C})$-valued gravitational gauge connection 1-form $\mathcal{A}$ on it which is a cross-section of its Ω^1 sub-bundle.

The present paper is organized as follows: in the next section we propose and discuss in some detail finitary versions of the principles of Equivalence and Locality of GR, as well as of their "corollaries," the principles of Local Relativity and Local Superposition, that are expected to be "operative" at the locally finite setting that we place our first step at modeling "finitary and causal Lorentzian quantum gravity" after "curving quantum causality by gauging a principal finsheaf

[22] That is to say, the dynamics of local quantum causality or "local quantum causal topology" and its symmetries.

[23] In the sense that "it avails itself to us as a topological space" by providing sufficient (but not necessary!) conditions for the definition of $\mathcal{A}_n$ which is the main dynamical variable in our theoretical scheme. See section 5.

of qausets."[24] In section 3 we review the algebraic model of flat quantum causality proposed in (Raptis, 2000b), namely, the qauset, and pronounce its characteristics to be subsequently localized or gauged. In section 4 we recall the topological finsheaves from (Raptis, 2000a), then we define finsheaves of qausets and their local causal symmetries. At the end of the section, a sound operational interpretation of finsheaves of qausets and their symmetries is given briefly, so that our theory is shown to have a strong philosophical support as well. In section 5 we suggest that for localizing or gauging and, as a result, curving quantum causality, a principal finsheaf of qausets having as structure group a finitary version of $SO(1,3)^{\uparrow}$, together with a discrete and local sort of a nonflat, spin-Lorentzian connection $\mathcal{A}_n$ on it, is an operationally sound model. $\mathcal{A}_n$ is then physically interpreted as a finitary, local, causal and quantal topological variable whose nonzero curvature stands for a finitary, causal, and quantal model of Lorentzian gravity. We conclude the paper by discussing the soundness of this finsheaf model of finitary and causal Lorentzian quantum gravity, as well as some physico-mathematical issues that derive from it.

2. PHYSICAL PRINCIPLES FOR FINITARY LORENTZIAN QUANTUM GRAVITY

In this section we commence our endeavor to model connection and its associated curvature in a curved finitary quantum causal setting by establishing heuristic physical principles that must be encoded in the very structure of our mathematical model[25] on which the dynamics of a locally finite quantum causality is going to be founded in section 5. The four physical principles to be suggested below will be seen to be the finitary and (quantum) causal analogues of the ones of Equivalence, Locality, as well as their "corollary" principles of Local Relativity and Local Superposition respectively, of GR which is formulated as a gauge theory in the principal bundle $\mathcal{P}$ over a differential manifold spacetime M. We have chosen these principles from the classical theory of gravity, because they show precisely in what way the latter is a type of gauge theory, and also because they will prepare the reader for our localization or gauging and concomitant curving of qausets in section 5.

The first physical principle from GR that we would like to adopt in our inherently granular scenario, so that curvature may be easily implemented and straightforwardly interpreted as gravity in such a finitary quantum causal context,

[24] As we said, the word "gauging" pertains to the aforementioned implementation of a nonflat gauge connection $\mathcal{A}_n$ on the finsheaf in focus.

[25] The "principal finsheaf of qausets with a nonflat finitary spin-Lorentzian connection on it," to be built progressively in the next three sections.

is that of equivalence (EP). We borrow from GR the following intuitively clear version of the EP:

Classical Equivalence Principle (CEP): The curved spacetime of GR is locally Minkowskian; thence flat. That is to say, the space tangent to every spacetime event is isomorphic to flat Minkowski space $\mathcal{M}$. As we mentioned in the introduction, in this sense GR may be thought of as SR made local or been pointwise (i.e., event-wise) gauged in M. Expressed thus the CEP effectively encodes Einstein's fundamental insight that locally the gravitational field $g_{\mu\nu}$ can be "gauged away" or be reduced to the constant and flat Minkowski metric $\eta_{\mu\nu} = diag(-1, +1, +1, +1)$ of SR[26] by passing to a locally inertial frame (Torretti, 1981).

What is important to emphasize in this formulation of the CEP is that in GR $\mathcal{M}$ assumes a local kinematical role in the sense that an isomorphic copy of it is erected, as some kind of fiber space, over each and every event of the differential manifold spacetime M, so that every individual fiber is physically interpreted as an independent (of the other $\mathcal{M}$-fibers) vertical world of spacetime possibilities along which the dynamically variable field of locality $g_{\mu\nu}$ can be reduced to the constant $\eta_{\mu\nu}$. It follows that the symmetries of gravity are the isometries of $\mathcal{M}$ localized; hence, one arrives at a gauged or localized version of the (orthochronous) Lorentz group as the invariances of GR. This motivates us to formulate the Classical Local Relativity Principle (CLRP) which, in a sense, is the local dynamical symmetry corollary of the CEP above:

Classical Local Relativity Principle (CLRP): The group of local (gauge) invariances of GR is isomorphic to the orthochronous Lorentz group $L^+ = SO(1, 3)^\uparrow$ of symmetries of the Minkowski space $\mathcal{M}$ of SR.

To summarize, the curved spacetime of GR may be modeled after the locally Minkowskian tangent vector bundle $TM := \cup_{x \in X \subset M} \mathcal{M}_x$—which is a sub-bundle of the dual of the $\mathbf{G}$-bundle $\mathcal{P}$ which has as continuous structure group $\mathbf{G} = SO(1, 3)^\uparrow$—together with a nonflat Lie algebra $\mathbf{g} = so(1, 3)^\uparrow \simeq sl(2, \mathbb{C})$-valued spin-Lorentzian Ω^1-section $\mathcal{A}$.

Since, as it was mentioned in the introduction, causets effectively encode the entire geometry of flat Minkowski space $\mathcal{M}$, they can be thought of as local kinematical structures representing the possible local causal relations in an otherwise curved spacetime of events. The CEP, modified to fit a finitary, causal, and curved situation like ours, reads.

[26] Since $\eta_{\mu\nu}(x)$ delimits the Minkowski lightcone at x for every $x \in M$, which, in turn, defines the local causal relations between events in the Minkowski space tangent to x, the gravitational potential $g_{\mu\nu}$ may be alternatively interpreted as "the dynamical field of local causality"—local causality being commonly known as "locality." Thus GR may be viewed as "the classical dynamical theory of locality."

Finitary Equivalence Principle (FEP): A finitary curved causal space is locally[27] a causet. Presumably, and on discrete locality grounds, it is the transitivity of causality as a partial order that must be renounced due to gravity (Finkelstein, 1988; Raptis, 1998, 2000b, in preparation-a).

In other words, a curved smooth spacetime, as a causal space, is not globally transitive; it is only locally and kinematically so.[28] Thus, the CEP may be restated as a correspondence or reduction principle: as the dynamically variable gravitational potential $g_{\mu\nu}$ reduces locally to the constant $\eta_{\mu\nu}$ in GR, causality becomes locally[29] the constant transitive partial order $\rightarrow$.[30] Equivalently, a curved finitary causal space, one having a causal relation not fixed to a globally transitive partial order, but with a dynamically variable local causality between its events, is only locally reducible to a transitive, flat "inertial causet." Thus, as $\mathcal{M}$ may be thought of as vertically extending, as an independent kinematical fiber space, over every event of the curved smooth spacetime manifold of GR, so an independent causet space may be thought of as being raised over every point event of a curved finitary causal space. Hence, the FEP almost mandates that a curved finitary causal space be modeled after a finsheaf (or a bundle) of causets (over a finitary spacetime)—with each independent causet being localized, so to speak, over the events of the finitary base space. As a matter of fact, and also due to the finitary principle of Locality that we will formulate shortly, we will see that a curved finitary causal space should be modeled after a finsheaf of qausets (not of transitive causets) for discrete locality's sake. Thus, some kind of "quantumness" will inevitably be infused *ab initio* into our model of the dynamics of finitary causality.[31] Before we give the Finitary Locality Principle and its "corollary," the Finitary Local Superposition Principle, we give the finitary analogue of the CLRP.

Finitary Local Relativity Principle (FLRP): The local invariance structure of a curved finitary causal space is a finitary version of L^+.

In a causal context, the work of Zeeman (1964, 1967) has shown that the symmetry structure L^+ of the flat Minkowski continuum $\mathcal{M}$, regarded as a causal space with a causality relation between its events modeled after a (globally) inertial partial order $\rightarrow$ which, in turn, derives from $\mathcal{M}$'s $\eta_{\mu\nu}$, is isomorphic to the

[27] Locality pending definition in our finitary context.

[28] That is to say, it is flat only in the "vertical" direction along each of the Minkowskian fibers of the curved covector bundle $\mathcal{P}$.

[29] As it was mentioned earlier, locality pending definition in our finitary scenario (see the principle of Finitary Locality below).

[30] With the CEP in mind, we may call "$\rightarrow$" "the inertial Minkowskian causality." In a curved causal space causality can only locally be the inertial partial order $\rightarrow$ (CEP).

[31] This infusion converts our scheme to a model of the dynamics of finitary *quantum* causality. See our formulation of the Finitary Local Superposition Principle below.

group **G** of causal automorphisms of $\mathcal{M}$.[32] In our case, and in view of the FEP, we infer that a finitary version of L^+, which we call $\mathbf{G}_n$, comprises the local relativity structure group of a curved finitary causal space.[33] Now, due to the local isomorphism mentioned in the introduction between the Lie algebras $\ell^+ = so(1, 3)^\uparrow$ and $sl(2, \mathbb{C})$ in the smooth **G**-bundle $\mathcal{P}$, we may alternatively say that $\mathbf{G}_n$ is the finitary version of the local relativity group $SL(2, \mathbb{C})$ of GR in its spinorial gauge-theoretic formulation (Ashtekar, 1986; Baez and Muniain, 1994; Bergmann, 1957). A similar local relativity group for a curved finitary quantum causal space was proposed in (Finkelstein, 1988),[34] whereas Selesnick (1994, 1998) found that $SL(2, \mathbb{C})$ is the result of a condensation of a quantum version of the classical binary alternative **2**—the permutation local relativity group of Finkelstein's reticular and curved quantum causal net.

In all the principles and remarks above, we mentioned the word "local" without having transcribed the notion of classical locality to a curved finitary causal scenario like ours. We do this now. The Classical Locality Principle (CLP) in GR may be wrapped up into the following assumption.

Classical Locality Principle (CLP): The spacetime of GR is modeled after a differential (i.e., C^∞-smooth) manifold M (Einstein, 1924).[35]

Since a locally finite causal model like ours does not involve (by definition) a continuous infinity of differentially (i.e., smoothly) separated events like the M above, the CLP on M may be translated in reticular causal terms to the following requirement.

Definition of Finitary Locality (DFL): In a causet, locality pertains to physical properties, to be interpreted as observables or dynamical physical variables, with "effective range of action or dynamical variation" restricted to empty Alexandrov

[32] Since the Alexandrov causal topology of $\mathcal{M}$ is defined by $\rightarrow$ (Alexandrov, 1956a,b, 1967), it follows that **G** is the group of causal homeomorphisms of $\mathcal{M}$ (Torretti, 1981).

[33] Then, $\mathbf{G}_n$ consists of the local causal homeomorphisms of the dynamically variable (and quantal) local causal topology of a curved finitary causal space. Since we plan to model the latter after a finsheaf of qausets whose local structure is (by definition) the qauset stalks over this curved finitary causal base space, $\mathbf{G}_n$ may be equivalently thought of as consisting of the group of homomorphisms (or automorphisms) of the quantal and causal incidence algebras respresenting these qauset stalks (Raptis, 2000b). (See also remarks at the end of this section on the significance of our choice to model the dynamics of a finitary quantum causality by a finsheaf of quasets.)

[34] We refer to the local SL_2 invariances of the dyadic cell of the net there.

[35] Thus, the CLP may be viewed as the requirement that all the dynamical laws of physics must be differential equations, or more intuitively, that local dynamical actions connect (influence) infinitesimally or "differentially" separated events living in the tangent space at each event of the smooth spacetime continuum. It follows that the CLP requires physical observables or dynamical variables to be modeled after (sections of) smooth differential forms (in $\mathcal{P}$), as mentioned in the introduction. Thus, by "the local structure of the curved spacetime manifold M" we mean "an event x and the space of directions tangent to it" (Raptis and Zapatrin, 2000). In the bundle $\mathcal{P}$ this pertains to its, Minkowskian by the CEP, fibers over each and every event x of its base spacetime manifold M.

sets.[36] Hence, we shall demand that the following physical principle be obeyed by our model of a curved finitary causal space.

Finitary Locality Principle (FLP): Dynamical relations on a causet $(X, \rightarrow)$ involve only finitary local observables.[37]

Some scholia on DFL and FLP are due here. Since in our reticular scheme we can assume no dynamical properties varying between infinitesimally (i.e., smoothly) separated events—as if an ether-like spacetime continuum serves as an inert connection medium between them—we may as well define local physical observables as the entities that vary between nearest neighboring events called "contiguous" from now on.[38] The FLP can be alternatively coined "the principle of contiguity in a finitary causal space" and it is the reticular analogue of the CLP of GR, which, in turn, as it was posited above, may be summarized to the assumption of a 4-dimensional differential manifold model for spacetime (Einstein, 1924).[39] Also, by the FEP above, we expect that in a curved finitary causal space gravity "cuts-off" the transitivity of inertial causality and restricts the latter to empty Alexandrov causal neighborhoods of contiguous events.

At this point it must be mentioned that the FLP, apart from seeming rather natural to assume, was somewhat "forced" on us by discrete topological and local quantum causal considerations. In more detail, it has been recently shown (Breslav *et al.*, 1999) that the generating relation ρ of the Rota topology of the incidence algebra Ω associated with a poset finitary substitute P of a continuous spacetime manifold as in (Sorkin, 1991) is the same as the one generating the finite poset topology of P if and only if one considers points in the Hasse diagram of the latter that are immediately connected by the partial order "$\rightarrow$" (i.e., "contiguous events").

[36] See (Bombelli *et al.*, 1987; Raptis, 2000b; Sorkin, 1990a,b, 1995) and the next footnote for a definition of these.

[37] Thus, only dynamical changes of observables between "nearest neighboring events" defining null Alexandrov sets in X (i.e., "$p, q \in X : (p \rightarrow q) \wedge (\nexists r : p \rightarrow r \rightarrow q)$," or in terms of the Alexandrov interval bounded by p and q, $A(p, q) := \{r : p \rightarrow r \rightarrow q\} = \emptyset$) are regarded as being physically significant. This principle is an explication of the definition of nonmediated (immediate) physical dynamical actions in the DFL above. Thus, by the DFL we anticipate the gravitational connection $\mathcal{A}$ in its finitary and causal version $\mathcal{A}_n$, which is supposed to be the main gravitational dynamical variable in our granular scheme, to be defined (as varying) on such immediate causal arrows. See section 5 for more on this.

[38] In the last footnote, p and q in the causet $(X, \rightarrow)$ are contiguous.

[39] Parenthetically, we mention that in this paper Einstein concludes that the smooth geometrical manifold model for spacetime, which is postulated up-front in GR for classical locality's sake, may be thought of as an inert and absolute ether-like background structure on which the whole theory of GR and the mathematical language that supports it, classical Differential Geometry, is erected. In view of his characteristic dissatisfaction with any theory that employs structures that are absolute and nondynamical, ultimately, "unobservable substances," and in view of the reticular, molecular picture of Nature that the quantum revolution brought about, we infer that Einstein could not have been content with the smooth manifold model for spacetime. Indeed, we find that this was the case (Einstein, 1936, 1956)—see quotations concluding the paper.

Then, if one interprets "→" in the finitary poset causally instead of topologically as in (Raptis, 2000b; Sorkin, 1995), and gives a cogent quantum interpretation to the structure of the causal Rota algebra associated with it as in (Raptis, 2000b; Raptis and Zapatrin, 2000), one is led to infer that the physically significant, because local, causal connections between events in a qauset are the contiguous, immediate ones; hence the FLP above. This was first anticipated by Finkelstein (1988). The FLP promotes this conjecture to a physical "axiom" or principle concerning the finitary dynamics of local (quantum) causality in a curved locally finite (quantum) causal space in the same way that in the differential manifold spacetime M of GR locality was "forced" on Einstein by M's own smoothness.

According to the finitary principles formulated above, we may say that in the same way that the CEP foreshadows a nontrivial connection $\mathcal{A}_n$ (and its associated curvature $\mathcal{F}_n$) in the smooth continuum—the main local dynamical variable of GR as a gauge theory in $\mathcal{P}$, so FLP, by "cutting-off" the transitivity property of "→," furnishes us with the crucial idea of how to model the dynamics of a contiguous (local) quantum causality in a curved finitary causal space, namely, one must define a nontrivial finitary connection (and its associated curvature) on a finsheaf of qausets over it. This connection, in turn, like its smooth spin-Lorentzian counterpart $\mathcal{A}$ on $\mathcal{P}$ over M respects local relativistic causality,[40] should somehow respect the local quantum causal connections in the qauset fibers.[41] This highlights and anticipates two very important aspects of the present paper:

(a) The finitary connection $\mathcal{A}_n$ (and its associated curvature) derives from the local algebraic structure of the finsheaf of qausets.[42] Thus, our scheme allows for a purely algebraic and local definition of connection (and its associated curvature) without reference to a background geometric base space[43] which will only serve as a surrogate host of $\mathcal{A}_n$ and which will have to be discarded, or at least be regarded as being physically insignificant, at the quantal level, only to be recovered as a fixed inert (nondynamical) geometrical structure at the classical limit of an inverse system of curved finsheaves of qausets.

[40] Since it preserves the Minkowski lightcone soldered (with origin) at each point-event x of M—the Minkowski lightcone in each fiber space $\mathcal{M}_x$ of $\mathcal{P}$.

[41] That is to say, it should respect the generating or "germ" relation $\vec{\rho}$ of the Rota quantum causal topology of the qauset stalk of the finsheaf in focus. We define $\vec{\rho}$ in the next section and germs of (continuous sections of) finsheaves of qausets in section 4.

[42] That is to say, from the algebraic structure of the quantal and causal incidence algebra stalks of the finsheaf in focus.

[43] This is in glaring contrast to the situation in the curved geometrical point set manifold M of GR where connection is intimately associated with a parallel transporter (of smooth tensor fields) along smooth finite spacetime curves, whereas its associated curvature $\mathcal{F}$ measures the anholonomy of such parallel transports around smooth finite spacetime loops (Göckeler and Schücker, 1990). Certainly, both are nonlocal geometric conceptions of $\mathcal{A}$ and its $\mathcal{F}$.

(b) A sheaf (and a nontrivial connection on it) is the "right" (i.e., the appropriate and natural) mathematical structure for modeling the dynamics (i.e., the curving) of local quantum causality, since, by definition, a sheaf is a local homeomorphism (Bredon, 1967; Mallios, 1998a; Raptis, 2000a), so that a $\mathbf{G}_n$-finsheaf of qausets by definition respects the reticular local quantum causal topology of the qauset stalks, while a non-flat $\mathbf{g}_n$-valued connection $\mathcal{A}_n$ on it effectively encodes the "local twisting" (curving) of these stalks relative to each other, thus it represents the dynamics of a locally finite quantum causality. We will return to these issues more analytically in the next three sections.

We close this section by giving the analogue of the kinematical Coherent Local Superposition Principle in (Finkelstein, 1988, 1991) for our finsheaves of qausets.

Finitary Local Superposition Principle (FLSP): Stalk-wise in a finsheaf of qausets the latter superpose coherently. It follows from the FLRP that the $\mathbf{g}_n$-valued connection $\mathcal{A}_n$ preserves this "stalk-wise quantum coherence or local superposition of qausets."[44]

In the next section we present an algebraic approach to flat (i.e., nondynamical or ungauged) reticular local quantum causality, whereas in sections 4 and 5 we motivate the finsheaf-theoretic point of view and we study a gauged, thus curved, principal finsheaf of qausets, respectively.

3. FINITARY SUBSTITUTES AND THEIR FLAT QUANTUM CAUSAL RELATIVES

In this section we motivate the modeling of qausets after incidence algebras *à la* Raptis (2000b), so as to prepare the reader for our representing the stalks of a finsheaf of qausets over some curved finitary causal space as such Rota algebras in section 5. The relevance of qauset theory to the problem of discrete Lorentzian quantum gravity is also discussed. In particular, we approach the issue of "discrete locality" or "finitary local causality" via qausets. We quote the main result from

[44] This is so since $\mathcal{A}_n$ takes values in the reticular (and quantal) algebra $\mathbf{g}_n$ of Rota algebra homomorphisms which, in turn, by the functorial equivalence between the category of finitary posets/poset morphisms (or its corresponding category of locally finite causets/causal morphisms) and the category of incidence Rota algebras/Rota homomorphisms (or its corresponding category of qausets/qauset homomorphisms) (Raptis and Zapatrin, 2000, in press; Stanley, 1986; Zapatrin, in press), it may be regarded as the reticular and quantal version of Zeeman's (1964) Lie algebra ℓ^+ of orthochronous Lorentz transformations (i.e., the infinitesimal causal automorphisms) of the Minkowski continuum $\mathcal{M}$ regarded as a flat inertial poset causal space. We will return to this remark in sections 4–6, but the upshot is that as a linear operator-valued map, $\mathcal{A}_n$ will preserve the local linear structure stalk-wise, hence, the local quantum coherence or quantum interference of qausets dwelling in these stalks.

(Raptis, 2000b) that qausets are sound models of a local and quantal version of the causets of (Bombelli *et al.*, 1987; Sorkin, 1990a,b, 1995) and use it as a theoretical basis to implement the FLP of the previous section, as well as to introduce the central physical idea for curving local quantum causality in section 5 by localizing or gauging a finsheaf of qausets (section 4), thus also realize the FEP of the previous section.

The topological discretization of continuous spacetime (Sorkin, 1991) has as its main aim the substitution of a continuum of events by some finitary, but topologically equivalent, structure. The latter is seen to be a T_0 poset. Such a finitary substitute for the continuous spacetime may be viewed as an approximation of its continuous counterpart, but one of physical significance, since it seems both theoretically and experimentally lame to assume a continuum as a sound model of what we actually experience (i.e., record in the laboratory) as "spacetime" (Raptis and Zapatrin, 2000). The theoretical weakness of such an assumption is the continuous infinity of events that one is in principle able to pack into a finite spacetime volume resulting in the unphysical infinities that plague classical and quantum field theory. The experimental weakness of the continuous model of spacetime is that it undermines the operational significance of our actual spacetime experiments, namely, the fact that we record a finite number of events during experimental operations of finite duration in laboratories of finite size; altogether, in experiments of finite spatiotemporal extent (Raptis and Zapatrin, 2000). Also, from a realistic or pragmatic point of view, our localizations (i.e., determinations of the loci) of events are coarse or "approximate" and inflict uncontrollable perturbations to the structure of spacetime,[45] thus our rough, because dynamically perturbing, measurements of events may as well be represented by open sets or "regions" about them (Breslav *et al.*, 1999; Butterfield and Isham, 2000; Raptis, 2000a; Raptis and Zapatrin, 2000, in press; Sorkin, 1991, 1995).

Of course, the discrete character of such finitary approximations of a continuous spacetime ties well with the reticular and finite characteristics that a cogent quantal description of spacetime structure ought to have. Thus, if anything, topological discretizations should prove useful in modeling the structure and dynamics of spacetime at quantum scales (Raptis and Zapatrin, 2000, in press). It must be stressed however that such a contribution to our quest for a sound quantum theory of gravity is not mandatory from the point of view of GR—the classical theory of gravity, since in the latter the topology of spacetime is fixed to that of a locally Euclidean manifold, while only the Lorentzian metric on it is assumed to be a dynamically variable entity. Effectively, $g_{\mu\nu}$ is the sole "observable" in GR. However, it seems rather *ad hoc* and unreasonably short sighted in view of the persisting and pestilential problem of the quantum localization of spacetime events to assume

[45] Even more so in our scenario where spacetime itself is assumed to be fundamentally a quantum system (Raptis and Zapatrin, 2000, in press).

that only the metric, but not the topological structure of the world, is subject to (quantum) dynamical fluctuations and variations. Such a theory of "spacetime foam," that is to say, of a dynamically fluctuating and in principle observable quantum spacetime topology, has been aired for quite some time now (Wheeler, 1964), and it is akin in spirit to the topological discretizations developed in (Sorkin, 1991), as well as to their quantal relatives in (Raptis and Zapatrin, 2000).[46] An attempt at an entirely algebraic description of quantum spacetime foam, akin to the finsheaf localizations of qausets to be worked out here, was recently proposed in (Raptis and Zapatrin, in press).

On the other hand, in view of the unphysical, nondynamical, nonrelativistic, space-like nature of the constant, 2-way, spatial connections between events that define the fixed locally Euclidean topology of the classical spacetime continuum M, there is an important affinity between our quest for a dynamical theory of local quantum causal topology and the problem of constructing a reasonable quantum theory of gravity. To understand this close relationship, we must change focus of enquiry from a theory of spatial Euclidean connections between geometric points to a more physical, because relativistic, temporal or causal spacetime topology between events as the quotation opening this paper suggests and as strongly advocated in (Finkelstein, 1988).

As it was mentioned in the previous section, in GR the gravitational potential, which is identified with the metric $g_{\mu\nu}$ of spacetime, may also be thought of as encoding complete information about the local causal relations between events. Thus, GR may also be interpreted as the classical dynamics theory of locality. It follows that a quantum theoresis of the dynamics of causal connections between spacetime events may lead to, if not just give us invaluable clues about, a classically conceived quantum theory of gravity—the quantization of the gravitational field $g_{\mu\nu}(x)$ of GR. In short, there probably is a way from a dynamical theory of local quantum causality to the graviton, but not the other way around (Bombelli *et al.*, 1987). A full fledged noncommutative topology for curved (i.e., dynamical) local quantum causality is rigorously formulated in the scheme-theoretic language of modern algebraic geometry and its categorical outgrowth, topos theory, in (Raptis, in press).

However, it must be stressed that it is quite clear, at least from a *gedanken* experimental point of view, why GR and QT are incompatible: the more accurately one may try to determine (i.e., localize) the spacetime metric, the more energy one must employ, the stronger the dynamical perturbations inflicted on it, the higher the uncertainty of its local determination.[47] Another way to say this is that we cannot distinguish or measure the proper pseudometric distance between spacetime

[46] See (Sorkin, 1995) for some discussion on this affinity.

[47] That is to say, the CEP on which GR is essentially based comes straight into conflict with the Uncertainty Principle on which QT is founded (Candelas and Sciama, 1983; Donoghue *et al.*, 1984, 1985; Sorkin, 1995).

events (via the gravitational potential $g_{\mu\nu}$) at a resolution higher than the Planck length ($l_P \approx 10^{-35}m$) without creating a black hole which, in return, "blurs" this separation of theirs.[48] This limitation alone is sufficient to motivate some kind of "topological foam" conception of spacetime at quantum scales (Raptis and Zapatrin, in press; Wheeler, 1964). An analogous incompatibility (of physical principles) that may hinder the development of a quantum theory of the dynamics of a finitary causality has not been foreseen yet. We hope that such a fundamental conflict of physical principles will be absent *ab initio* from an innately locally finite dynamical theory of quantum causality, or at least from the kinematics of such a theory like the one that we will propose in section 5.

This lengthy prolegomenon to the introduction of the flat qausets in (Raptis, 2000b) highlights two important aspects of our present endeavor: first, our locally finite, and to be gauged subsequently, qausets may evade *ab initio* the infinities of QGR on a smooth manifold, and second, as quantum causal–topological structures, they grope with the problem of the structure and dynamics of spacetime at quantum scales at a level deeper than QGR proper which is supposed to study solely the quantum aspects of the dynamics of the metrical structure of the world, because we have seen already at the classical level that causality, as a partial order, and its morphisms, determine the conformal geometric structure of flat Minkowski space and its symmetries (Alexandrov, 1956a,b, 1967; Bombelli and Meyer, 1989; Robb, 1914; Sorkin, 1990a,b; Zeeman, 1964, 1967). After all, as Bombelli *et al.* successfully observed in (1987), it is such a model for events and their causal relations that uniquely determines spacetime as a 4-dimensional, continuous (C^0), differential (C^∞-smooth), and Lorentzian (i.e., of signature ± 2) metric manifold.

We commence our brief review of qausets by first recalling very briefly some important facts about finitary substitutes for continuous spacetime topology (Raptis, 2000a,b; Raptis and Zapatrin, 2000, in press; Sorkin, 1991). Let X be a bounded region in a continuous spacetime manifold M[49] and $\mathcal{U} = \{U\}$ a locally finite open cover of it.[50] Any two points x and y of X are said to be indistinguishable with respect to its locally finite open cover $\mathcal{U}$ if $\forall U \in \mathcal{U} : x \in U \Leftrightarrow y \in U$. Indistinguishability with respect to the subtopology $\mathcal{T}(\mathcal{U})$[51] of X is an equivalence relation on the latter's points and is symbolized by $\overset{\mathcal{U}}{\sim}$. Taking the quotient $X/\overset{\mathcal{U}}{\sim} =: F$

[48] This is the arch paradox of event localization that makes the conception of a quantum theory of gravity hard even in principle: the more accurately we try to localize spacetime events, the more we blur them, so that our sharpest determinations of them can be modeled after coarse, rough, fuzzy, "dynamically fluctuating" open neighborhoods about them as in (Breslav *et al.*, 1999; Raptis, 2000a, 2001; Raptis and Zapatrin, 2000, in press; Sorkin, 1991, 1995; Zapatrin, 1998).

[49] By "bounded" we mean "relatively compact" (i.e., a region whose closure is compact). By "continuous" we mean the C^0 aspects of classical spacetime (i.e., its features as a topological manifold).

[50] That is to say, every point event x in X has an open neighborhood $O(x)$ that meets only a finite number of open sets U in $\mathcal{U}$.

[51] $\mathcal{T}(\mathcal{U})$ consists of arbitrary unions of finite intersections of the open sets in the cover $\mathcal{U}$.

results in the substitution of X by a space F consisting of equivalence classes of its points, whereby two points in the same equivalence class are covered by (i.e., belong to) the same, finite in number, open neighborhoods U of $\mathcal{U}$, thus are indistinguishable by (our coarse observations in) it.

Let x and y be points belonging to two distinct equivalence classes in F. Consider the smallest open sets in the subtopology $\mathcal{T}(\mathcal{U})$ of X containing x and y respectively given by: $\Lambda(x) := \cap\{U \in \mathcal{U} : x \in U\}$ and $\Lambda(y) := \cap\{U \in \mathcal{U} : y \in U\}$. Define the relation $\rightarrow$ between x and y as follows: $x \rightarrow y \Leftrightarrow \Lambda(x) \subset \Lambda(y) \Leftrightarrow x \in \Lambda(y)$. Then assume that $x \overset{\mathcal{U}}{\sim} y$ in the previous paragraph stands for $x \rightarrow y$ and $y \rightarrow x$;[52] write $x \leftrightarrow y$. $\rightarrow$ is a partial order on F and the continuous X has been effectively substituted by the finitary F which is a T_0 topological space having the structure of a poset (Sorkin, 1991). Sorkin uses the finitary topological and partial order-theoretic languages interchangeably exactly due to this equivalence between T_0 finitary substitutes and posets. For future purposes we may distill this to the following statement: in (Sorkin, 1991) a partial order is interpreted topologically. We shall call it "topological partial order" and the poset encoding it "topological poset" (Raptis, 2000b).

One can show that topological posets have an equivalent representation as simplicial complexes if instead of using Sorkin's "equivalence algorithm" above, one uses Alexandrov's "nerve construction" (Alexandrov, 1956a,b, 1961; Raptis and Zapatrin, 2000; Zapatrin, in press). In the nondegenerate cases, the posets associated with Alexandrov nerves and those produced by Sorkin's algorithm yielding Fs from X relative to $\mathcal{U}$s, are the same, so that both are "topological posets" according to our denomination of the Fs (Raptis, 2000b). In fact, the correspondence between the poset category consisting of topological posets/poset morphisms obtained from Sorkin's algorithm and the poset category of simplicial complexes/simplicial maps obtained from Alexandrov's construction, is functorial (Raptis and Zapatrin, 2000; Zapatrin, in press).

In (Raptis and Zapatrin, 2000) an algebraic representation of topological posets was presented using the so-called Rota incidence algebras associated with posets (Rota, 1968). The Rota incidence algebra Ω of a poset P was defined there by using Dirac's quantum ket-bra notation as follows:

$$\Omega(P) = \text{span}\{|p\rangle\langle q| : p \rightarrow q \in P\},$$

with product between two of its ket-bras given by:

$$|p\rangle\langle q| \cdot |r\rangle\langle s| = |p\rangle\langle q \mid r\rangle\langle s| = \langle q \mid r\rangle \cdot |p\rangle\langle s| = \begin{cases} |p\rangle\langle s| & \text{if } q = r \\ 0 & \text{otherwise} \end{cases}$$

Evidently, for the definition of the product in Ω, the transitivity of the partial order $\rightarrow$ in P is used. $\Omega(P)$, defined thus, is straightforwardly verified to be an

[52] That is to say, x and y have the same smallest open neighborhood about them in $\mathcal{T}(\mathcal{U})$.

associative algebra.[53] When P is a finitary topological poset in the sense of Sorkin (1991), its associated incidence algebra is called "topological incidence algebra" (Raptis, 2000b).

We may define purely algebraically a topology on any incidence algebra Ω associated with a poset P by considering its primitive spectrum $\mathcal{S}$ consisting of (equivalence classes of) its irreducible representations (Zapatrin, 1998), whose kernels are primitive ideals in it, in the following way according to Breslav *et al.* (1999). With every point p in P the following ideal in Ω is defined as:

$$I_p = \text{span}\{|q\rangle\langle r| : |q\rangle\langle r| \neq |p\rangle\langle p|\},$$

so that the Rota topology of $\Omega(P)$ is generated by the following relation ρ between "points" I_p and I_q in its primitive spectrum $\mathcal{S}$:

$$I_p \rho I_q \Leftrightarrow I_p I_q (\neq I_q I_p) \overset{\neq}{\subset} I_p \cap I_q.$$

It has been shown that the Sorkin topology of a topological poset P is the same as the Rota topology of its associated topological incidence algebra $\Omega(P)$ exactly when the generating relation ρ for the latter is identified with the transitive reduction $\overset{*}{\to}$ of the partial order arrows $\to$ in P (Breslav *et al.*, 1999; Raptis, 2000b).[54] This means essentially that the "germ relations" for the Rota topology on the algebra Ω associated with the finitary topology P are precisely the immediate arrows $\overset{*}{\to}$ in the latter topological poset which, in poset parlance, are called "the covering relations of the poset". This is an important observation to be used shortly in order to define in a similar way the germs of quantum causal relations in a qauset with respect to which finsheaves of the latter will be defined in the next section as structures that preserve precisely these local quantum causal topological "germ relations."

To this end we give the definition of qausets. First, a causet is defined in (Bombelli *et al.*, 1987) as "*a locally finite set of points endowed with a partial order corresponding to the macroscopic relation that defines past and future*." Local finiteness may be defined as follows: use $\to$ of a poset P, interpreted now as a causal relation on the set of vertices of P, to redefine $\Lambda(x)$ for some $x \in P$

[53] The associativity of the product of the incidence algebra Ω is due to the transitivity of the partial order $\to$ of its associated poset P. As we saw in section 2, it is precisely the latter property of causality, when modelled after the globally inertial $\to$, that is regarded as being responsible for the flatness of Minkowski space determined by $\to$. It follows that a localization or gauging of causets and their corresponding qausets in order to curve them, by providing a connection on a principal finsheaf of theirs, will "cut-off" the transitivity of the causets and the associativity of their corresponding qausets, and will restrict it locally (i.e., stalk-wise) in the finsheaf thus implement the FEP of section 2.

[54] That is to say, I_p is ρ-related to I_q if and only if $(p \overset{*}{\to} q) \Leftrightarrow [(p \to q) \wedge (\nexists r : p \to r \to q)$; $p, q, r \in P]$ (i.e., only for immediately connected or contiguous vertices in P).

as $\Lambda(x) = \{y \in P : y \rightarrow x\}$, and dually $V(x) = \{y \in P : x \rightarrow y\}$. $\Lambda(x)$ is the "causal past" of the event x, whereas $V(x)$ its "causal future." Then, local finiteness requires the so-called Alexandrov set or "causal interval" $A(x, y) := V(x) \cap \Lambda(y)$ to be finite for all $x, y \in P$ such that $x \in \Lambda(y)$. In other words, only a finite number of events "causally mediate" between any two events x and y, with $x \rightarrow y$, of the causet P. In a sense, the finitarity of the topological posets translates by Sorkin's semantic switch to the local finiteness of causal sets, although it must be stressed that the physical theories that they support, the discretization of topological manifolds in (Sorkin, 1991) and causet theory *per se* in (Bombelli *et al.*, 1987; Sorkin, 1995) respectively, are quite different in motivation, scope and aim (Raptis, 2000b; Sorkin, 1990a,b, 1991, 1995).

At the same time, it was Sorkin who first insisted on a change of physical interpretation for the partial order $\rightarrow$ of finitary posets P "*from a relation encoding topological information about bounded regions of continuous spacetimes, to one that stands for the relation of causal succession between spacetime events*" (Sorkin, 1995). In (Raptis, 2000b), this fundamental semantic switch was evoked to reinterpret the incidence algebras associated with the finitary posets in (Raptis and Zapatrin, 2000) from topological to causal. Thus, causal incidence algebras were defined as the Ωs associated with finitary posets P when the latter are interpreted as causets *à la* Bombelli *et al.* (1987). Of course, in our pursuit of a cogent quantum theory of the dynamics of causality and, *in extenso*, of gravity, such a change of physical meaning of finitary partial orders from spatial/choro-logical/topo-logical to temporal/chrono-logical/causal is very welcome for the reasons briefly given earlier in this section.

Finally, in (Raptis, 2000b) the quantum physical interpretation given to topological incidence algebras in (Raptis and Zapatrin, 2000) was also given directly to causal incidence algebras. In effect, the quantal interpretation of the causal incidence algebras rests essentially on the fact that in the new Rota algebraic environment the causal arrows of the causets from which these qausets derive can coherently superpose with each other—an operation that is prominently absent from the respective "classical" causets of Bombelli *et al.* (Raptis, 2000b; Raptis and Zapatrin, 2000, in press). Totally, qausets were defined as the causally and quantally interpreted Rota incidence algebras associated with poset finitary substitutes of continuous spacetimes. It follows that the generator $\stackrel{*}{\rightarrow}$ of topological relations in the topological posets of Sorkin becomes the germ $\vec{\rho}$ of quantum causal relations in qausets.[55] Its interpretation is as "immediate quantum causality"[56] and it is exactly due to its natural Rota algebraic representation that qausets are sound

[55] Due to its causal instead of topological meaning, we are going to write $\vec{\rho}$ instead of ρ from now on for this local quantum causal topological variable.

[56] As we said, the epithet "quantum" refers precisely to the possibility for coherent quantum superpositions of the causal arrows of $\vec{P}$ in its associated incidence algebra $\vec{\Omega}(\vec{P})$.

models of quantum causal spaces (Finkelstein, 1988; Raptis, 2000b). $\vec{\rho}$ is the algebraic correspondent in the causal incidence algebra $\vec{\Omega}$ of the immediate causal or contiguous or covering relation $\stackrel{*}{\rightarrow}$ of its associated causet $\vec{P}$ (Breslav *et al.*, 1999; Raptis, 2000b; Raptis and Zapatrin, 2000).

The immediate quantum causality represented by $\vec{\rho}$ in the incidence algebra associated with a causet is ideal for implementing the FLP of the previous section. In particular, it explicitly shows that the physically significant, because local, quantum causality is the relation $\stackrel{*}{\rightarrow}$ between immediately separated events in a finitary spacetime X (Finkelstein, 1988; Raptis, 2000b). Its nonlocal (Finkelstein, 1988) transitive closure, the partial order $\rightarrow$ in the associated causet $\vec{P}$, generates $\vec{P}$'s (globally) inertial Minkowskian causal topology which, being a finitary poset, essentially determines a locally finite version of flat Minkowski space and its global orthochronous Lorentz symmetries (Alexandrov, 1956a,b, 1967; Robb, 1914; Zeeman, 1964, 1967). This is spacetime, as a classical causal space, ungauged.

It follows that in order to curve qausets, a gauged or localized version of $\vec{\rho}$ must be employed, that is to say, we should consider a dynamical local quantum causal connection relation that only locally (i.e., event-wise) reduces to a transitive partial order—the inertial Minkowskian causality of a reticular and quantal Minkowski space (as a qauset) according to the FEP. In turn, this means that only the transitive reduction $\stackrel{*}{\rightarrow}$ of the flat global inertial causality $\rightarrow$ will be the physically significant local dynamical variable in a curved finitary quantum causal space. We will model this conjecture by a nonflat connection $\mathcal{A}_n$ on a finsheaf of qausets in section 5.

We conclude the present section by discussing briefly two relatively important aspects of qausets, one physical, the other mathematical.

The physical aspect of qausets pertains to their operational significance. Although the operational soundness of quantum discretized spacetimes has been fairly established (Raptis and Zapatrin, 2000) in that we have a sound and pragmatic operational interpretation of quantal topological incidence algebras, we still lack such an account for qausets. Now, GR's operational significance can be summarized in the following: $g_{\mu\nu}(x)$, which mathematically represents the local gravitational potential, is supposed to encode all the information about our local experimental tampering with spacetime events via synchronized clocks and equicalibrated rulers so that, in principle, from the data of such a local experimental activity, one can construct the metric tensor at a neighborhood of an event. In such an operational account, there is little room left for a "passive" realistic interpretation of the gravitational field as an independent entity or "real substance out there" whose interaction with our instruments yields readings of events. The operational approach is in an important sense more active in that it entails that spacetime attributes are extracted from "it" by our very experimental actions on (i.e., our planned, controlled and in principle reproducible observations of) "it."

Also, this seems to be more in accord with the observer-dependent conception of physical reality that QT supports (Finkelstein, 1996).

For the causets of Bombelli *et al.* (1987) and Sorkin (1990a,b), Sorkin (1995) contended that an operational interpretation is rather unnatural and lame. On the other hand, in view of the algebraic structure of qausets and the sound quantum-operational interpretation *à la* Raptis and Zapatrin (2000) that their topological counterparts were given, and because as we mentioned in section 2 the local field of gravity $g_{\mu\nu}(x)$ can also be interpreted as the dynamical field of the local causal topology of spacetime, we still hope for a sound operational interpretation of them. At the end of the next section we present our first attempt at a sound operational interpretation of the locally finite quantum causality encoded in qausets based on the analogous operational meaning of the finitary poset substitutes of continuous spacetimes and their incidence algebras (Raptis and Zapatrin, 2000; Sorkin, 1991, 1995). A more thorough presentation of the operational character of qausets will be given in (Raptis, in preparation-b).

The mathematical aspect of qausets that we would like to discuss next is their differential structure. Recently, there has been vigorous research activity on studying differential calculi on finite sets and the dynamics of networks (Dimakis *et al.*, 1995), as well as on defining some kind of discrete Riemannian geometry on them (Dimakis and Müller-Hoissen, 1999). The main result of such investigations is that with every directed graph a discrete differential calculus may be associated. It follows that for the locally finite posets underlying qausets $\vec{\Omega}$ (i.e., the causets $\vec{P}$ associated with them), which are also (finitary) digraphs, there is a discrete differential calculus associated with them (Raptis and Zapatrin, 2000, in press; Zapatrin, in press). In this sense, but from a discrete perspective, a partial order determines not only the topological (C^0), but also the differential (C^∞) structure of the spacetime manifold with respect to which the Lorentzian $g_{\mu\nu}$, which is also determined by causality as a partial order,[57] is then defined as a smooth field (Bombelli *et al.*, 1987).

However, as we noted in the introduction, the Kähler-Cartan type of discrete differential operator d defined in such calculi on finite sets is a flat sort of connection (Mallios, 1998a). This is not surprising, since the underlying finite spacetime X is taken to be a structureless point set—in a sense, a kind of disconnected, noninteracting dust. All the digraphs supporting such calculi are assumed to be transitive, so that if some causal interpretation was given to their arrows, by our heuristic principles of section 2 concerning the relation between an inertial transitive causality and flatness, their corresponding differential calculi should be flat as well.[58] This is the "curvature problem" alluded to in the introduction. To

[57] At least locally in a curved spacetime (see section 2).

[58] That is to say, the differential operators defining such calculi are flat connections in the sense of Mallios (1998a).

evade it, in section 5 we straightforwardly gauge (a finsheaf of) qausets so that a nonflat connection $\mathcal{A}_n$ is naturally defined on them. The physical interpretation of such a gauging of the d of flat qausets to the $D = d + \mathcal{A}$ of the curved finsheaf of qausets, will be the first essential step towards a finitary, causal and quantal version of Lorentzian gravity.

In the next section we recall the finsheaves from (Raptis, 2000a). Our principal aim is to bring forth the sense in which a finsheaf of continuous maps over Sorkin's topological posets substitutes the sheaf of C^0-topological observables over a continuous spacetime manifold, then try to "read" a similar physical meaning for a finsheaf of qausets, namely, that they model finitary and quantal replacements of the causal relations between events in a bounded region of flat Minkowski space $\mathcal{M}$, as well as the causal nexus of C^∞-smooth fields in such a region of the smooth differential manifold $\mathcal{M}$. At the same time, the finsheaves of their continuous symmetries may be thought of as reticular analogues of the continuous orthochronous Lorentz topological Lie group manifold $SO(1, 3)^\uparrow$. Thus, in such a scenario, not only the operational significance of our own pragmatic finitary "perceptions" of spacetime structure and its dynamics will be highlighted, but also the operational meaning of our rough and dynamically perturbing determinations of its symmetries—them too to be subsequently gauged.

4. FINITARY SPACETIME SHEAVES AND THEIR FLAT QUANTUM CAUSAL DESCENDANTS

In (Raptis, 2000a), a finsheaf S_n of continuous functions on a bounded region X of a topological spacetime manifold M was defined as the sheaf of sections of continuous maps on X relative to its covering by a locally finite collection of open subsets of M. Since, as we saw in the previous section, for every such finitary open cover $\mathcal{U}_n$ of X a finitary topological poset F_n was defined and seen to effectively substitute X, the aforementioned sheaf can be thought of as having F_n as base space. Thus, we write $S_n(F_n)$ for such a finsheaf (Raptis, 2000a). Indeed, S_n was seen to have locally the same finite poset topology as its base space F_n,[59] hence its qualification as a sheaf (Bredon, 1967; Mac Lane and Moerdijk, 1992; Mallios, 1998a).

Now, as we briefly alluded to in the introduction, the essential result from (Raptis, 2000a), and the one that qualifies finsheaves as sound reticular approximations of the continuous spacetime observables on X, is that an inverse system of finsheaves has an inverse limit topological space that is homeomorphic to $S(X)$—the sheaf of continuous functions on X, in the same way that in (Sorkin, 1991) an inverse system of finitary poset substitutes of X was seen to "converge" to a space that is homeomorphic to the continuous topological manifold X itself.

[59] Technically speaking, S_n was locally homeomorphic to the finitary topological poset F_n of (Sorkin, 1991).

To define finsheaves of qausets, we adopt from (Raptis and Zapatrin, 2000) the association with every poset finitary substitute F_n of a bounded spacetime region X, of a Rota incidence algebra $\Omega(F_n)$, as it was shown in the previous section. As F_n is a topological poset, its associated Ω_n is a topological incidence algebra (Raptis, 2000b). As we noted in the previous section, to get the qauset $\vec{\Omega}_n$ from Ω_n, we "causalize" and "quantize" it *à la* Raptis (2000b). As a result of such a causalization, we write $\vec{\rho}$ for the generating relation of $\vec{\Omega}_n$'s (quantum) causal topology in the same way that ρ in the previous section was seen to be the generator of Ω's "spatial" Rota topology. The significance of $\vec{\rho}$ is (quantum) causal, while of ρ only topological.

Finsheaves of qausets are then defined to be objects $\vec{S}_n := \vec{\Omega}_n(\vec{F}_n)$, whereby the local homeomorphism between the base causal set $\vec{F}_n$ and the algebra $\vec{\Omega}_n$ is now given, in complete analogy to the finsheaf $S_n(F_n)$ of topological posets in (Raptis, 2000a), as $p \stackrel{*}{\rightarrow} q \Leftrightarrow I_p \vec{\rho} I_q$, $(p, q \in \vec{F}_n, I_p, I_q \in \vec{\mathcal{S}}(\vec{\Omega}_n))$. As it was mentioned in the previous section, the Sorkin poset topology on the topological F_n, obtained as the transitive closure of the immediate contiguity relation $\stackrel{*}{\rightarrow}$ between its vertices, is the same as the Rota topology of its associated topological incidence algebra Ω_n generated by ρ, only now these relations have a directly causal/temporal rather than a topological or "purely spatial" significance (Raptis, 2000b; Sorkin, 1995).

Also, in the same way that $S_n(F_n)$ was seen to be the finsheaf of continuous maps on the F_n obtained from X with respect to its locally finite open cover $\mathcal{U}_n$ and generated by its (germs of) continuous sections (Raptis, 2000a), we may similarly consider $\mathbf{G}_n := \mathcal{L}_n(\vec{\Omega}_n)$ to be the finsheaf of local (quantum) causal (auto) morphisms of $\vec{\Omega}_n$. We may call $\mathbf{G}_n$ "the finitary spacetime transformation sheaf adjoint to $\vec{S}_n$."[60] The (germs of) continuous sections of this sheaf are precisely the maps that preserve the local (quantum) causal topology $\vec{\rho}$ of $\vec{\Omega}_n$, so that by the definition of the latter they are the $\vec{\Omega}_n$-homomorphisms "restricted" to the primitive ideals I_p and I_q in them—the Gel'fand "point events" of the qauset $\vec{\Omega}_n$[61] which is the finitary base space of the finsheaf $\mathbf{G}_n$.

The finsheaf $\mathbf{G}_n$ consists of the local causal homeomorphisms $\vec{\lambda} n$ of the local (quantum) causal topology (generated by) $\vec{\rho}$ of the qauset $\vec{\Omega}_n$ which, by the

[60] $\mathbf{G}_n$ is a group sheaf with carrier or representation or more commonly known as "associated" sheaf that of qausets $\vec{S}_n$. The proper technical name for $\mathbf{G}_n$ is "principal sheaf with structure group $\mathcal{L}_n$" although, as we also mentioned in the introduction, we use the latter name for the pair $(\vec{S}_n, \mathbf{G}_n)$.

[61] This topological interpretation of the primitive ideals of an incidence algebra Ω associated with a finitary poset substitute F in (Sorkin, 1991) as "space points," comes from the Gel'fand "spatialization procedure" used in (Breslav *et al.*, 1999; Zapatrin, 1998), whereby, the point vertices of the poset substitute F of X were corresponded to elements of the primitive spectrum $\mathcal{S}$ of its associated incidence algebra Ω which, in turn, are the kernels of (equivalence classes of) the irreducible representations of $\Omega(F)$. In our causal version $\vec{\Omega}$ of Ω, the primitive spectrum of the former is denoted by $\vec{\mathcal{S}}$ and its points (i.e., the primitive ideals of $\vec{\Omega}_n$) are interpreted as "coarse spacetime events"—they are equivalence classes of X's point events relative to our pragmatic observations $\mathcal{U}_n$ of them of limited power or energy of resolution (Cole, 1972; Raptis, 2000a; Raptis and Zapatrin, 2000).

discussion in section 2, constitutes the finitary version of the orthochronous Lorentz group L^+. Hence $\mathbf{G}_n$ may be thought of as the finitary substitute of the continuous Lie group manifold L^+ which, due to the (local) quantal character of the qausets in $\vec{\Omega}_n$, also inherits some of the latter's "quantumness" in the sense that since qausets coherently superpose with each other locally according to the FLSP of section 2, so will their symmetry transformations. This is in accord with Finkelstein's insight that if spacetime is to be regarded as being fundamentally a quantum system, then so must be its structure symmetries (Finkelstein, 1996). Thus, the finsheaf $\vec{S}_n$, together with its adjoint $\mathbf{G}_n$ of its local symmetries, constitute a principal $\mathbf{G}_n$-finsheaf of qausets and their finitary local causal (and quantal) homeomorphisms.

We may denote this principal finsheaf either by $\vec{\mathcal{M}}_n := \mathbf{G}_n(\vec{S}_n)$, or more analytically by $\vec{\mathcal{M}}_n := (\vec{F}_n, \vec{\Omega}_n, \mathcal{L}_n)$,[62] with the corresponding local homeomorphisms defining them as finsheaves (with a quantum causal topological interpretation) being denoted as $\vec{s}_n : (\vec{F}_n, \overset{*}{\rightarrow}) \to (\vec{\Omega}_n, \vec{\rho})$ and $\vec{\lambda}_n : (\vec{\Omega}_n, \vec{\rho}) \to (\vec{S}_n, \mathbf{g}_n)$—where the reticular local causal homeomorphism $\vec{\lambda}_n$ corresponds a $\vec{\rho}$-preserving map to an element in the reticular Lie algebra $\mathbf{g}_n$ of the structure group $\mathbf{G}_n = L_n^+$ of the $\mathbf{G}_n$-finsheaf $\vec{\mathcal{M}}_n$.[63]

The main conjecture in this paper, briefly mentioned at the end of (Raptis, 2000a) and in the introduction, and not to be analytically proved here, is that an inverse system $\mathcal{K}$ of the $\mathbf{G}_n$-finsheaves of qausets $\vec{\mathcal{M}}_n$ "converges" to the classical flat Minkowskian $\mathbf{G}$-sheaf $(X \subset \mathcal{M}, \Omega, d, L^+)$, where X is a bounded region in the smooth, flat Minkowski manifold $\mathcal{M}$, which serves as the base space for the sheaf of smooth differential forms Ω on it. This sheaf has as stalks over X's point events isomorphic copies of the $\mathbb{Z}$-graded module of Cartan exterior differential forms $\Omega := \Omega^0 \oplus \Omega^1 \oplus \Omega^2, \ldots, d$ is the nilpotent and flat Kähler-Cartan connection on the sheaf effecting (sub)sheaf morphisms $d : \Omega^i \to \Omega^{i+1}$ in the differential triad (X, Ω, d) (Mallios, 1998a), whereas L^+ is the continuous structure group of the sheaf consisting of the global orthochronous Lorentz transformations of $\mathcal{M}$.[64] Heuristic arguments supporting this conjecture are:

(a) The topological (i.e., C^0) structure of (X of) $\mathcal{M}$ as a topological manifold arises as the limit space of an inverse system of finitary incidence algebras

[62] The symbol "$\vec{\mathcal{M}}_n$" for "$\mathbf{G}_n(\vec{S}_n)$" will be explained shortly.

[63] $\mathbf{g}_n$ is the finitary version of the Lie algebra ℓ^+ of the orthochronous Lorentz group $\mathbf{G}_n = L^+ := SO(1, 3)^\uparrow$ whose algebraic structure is supposed to respect the "horizontal" reticular causal topology of $\vec{\Omega}n$ which is generated by $\vec{\rho}$—"the germ of the local quantum causal topology" of the qauset stalks $\vec{\Omega}_n$ of $\mathbf{G}_n$'s associated finsheaf $\vec{S}_n$ (Raptis, 2000a).

[64] This description of the sheaf $(\mathcal{M}, \Omega, d, L^+)$ makes it the $\mathbf{G}$-sheaf-theoretic analogue of a $\mathbf{G}$-bundle of exterior forms having as base space the flat Minkowski differential manifold $\mathcal{M}$, as fibers modules of smooth Cartan forms on $\mathcal{M}$, as flat generalized differential (i.e., connection) structure the nilpotent Kähler-Cartan differential d, and as structure group the orthochronous Lorentz group L^+. One may regard this sheaf as the mathematical structure in which classical as well as quantum field theories are formulated in the absence of gravity.

$\Omega_n(F_n)$, now topologically interpreted, as shown in (Raptis and Zapatrin, 2000; Sorkin, 1991). It can also be determined from the causally interpreted incidence algebras $\vec{\Omega}_n(\vec{F}_n)$ as suggested in (Bombelli *et al.*, 1987).

(b) The differential (i.e., C^∞-smooth) structure of (X in) $\mathcal{M}$ as a differential manifold supporting fibers of modules Ω of Cartan's exterior forms, arises as the limit space of an inverse system of finitary incidence algebras $\Omega_n(F_n)$, since the latter have been seen to be discrete differential manifolds in the sense of (Dimakis and Müller-Hoissen, 1999; Raptis and Zapatrin, 2000, in press; Zapatrin, in press). In fact, as Dimakis *et al.* (1995) show, the discrete differential structure of such discrete differential manifolds also determines their (finitary) topology.[65] The differential structure of $\mathcal{M}$ can also be determined from the causally interpreted incidence algebras $\vec{\Omega}_n(\vec{F}_n)$ as suggested in (Bombelli *et al.*, 1987).

(c) A reticular analogue of the Minkowski metric $\eta_{\mu\nu} = diag(-1, +1, +1, +1)$ on $\mathcal{M}$ is determined by the causal incidence algebra $\vec{\Omega}_n$ associated with the causal set $\vec{F}_n$ also as suggested in (Bombelli *et al.*, 1987).[66] It must be emphasized however that in order to determine an indefinite Lorentzian spacetime metric such as $\eta_{\mu\nu}$, the causally interpreted finitary incidence algebras must be used, not the topological ones. This is because, as it was shown in (Dimakis and Müller-Hoissen, 1999), the discrete metric g that is naturally defined on a discrete differential manifold such as the finitary topological incidence algebra of (Raptis and Zapatrin, 2000), is positive definite (Riemannian), rather than indefinite (pseudo-Riemannian or Lorentzian). This is the "signature problem" alluded to in the introduction. The solution of the "signature problem" by using causets instead of topological posets justifies Finkelstein's (1988) and Sorkin's (1995) demand for a physical causal or temporal topology instead of an unphysical spatial one, as we emphasized in the previous section.

(d) The Kähler-Cartan differential operator d that defines the differential structure of $\mathcal{M}$ in (b) is a flat connection on the differential triad sheaf $(\mathcal{M}, \Omega, d)$ (Mallios, 1998a), as it is expected to be for the flat Minkowski base space $\mathcal{M}$. In (Dimakis and Müller-Hoissen, 1999), a connection ∇ and its associated curvature $R := -\nabla^2$ are defined, and compatibility conditions between ∇ and the definite metric g are given that make the connection a metric one. However, since as it was mentioned in (c), g is a positive definite metric, ∇ will not do, for we are looking for a pseudo-Riemannian (Lorentzian) connection on our finsheaves of qausets. Furthermore, as it was also shown in (Dimakis and Müller-Hoissen, 1999), for

[65] That is to say, "*differentiability implies continuity*"—the classic motto in university Calculus.

[66] As we mentioned in the introduction, the work of Robb (1914) already shows that causality as a partial order determines a Lorentzian metric up to its determinant (spacetime volume measure). See also (Bombelli and Meyer, 1989; Sorkin, 1990a,b).

the most general (universal) discrete differential calculus on a discrete differential manifold, ∇ reduces to the flat (because nilpotent) Kähler-Cartan differential d, so that there can be no discrete (not even positive definite, i.e., Riemannian) "gravity" on it. This is the "flatness problem" alluded to in the introduction. The flatness problem will be tackled in the next section by a straightforward localization or gauging of qausets in their finsheaves.

(e) Finally, for the sheaf of global orthochronous Lorentz transformations that we expect to arise as the group sheaf (Mallios, 1998a) of (global) symmetries of its adjoint or associated flat Minkowskian sheaf $(\mathcal{M}, \Omega, d)$ from an inverse system of finsheaves $\mathbf{G}_n = \mathcal{L}_n(\vec{\Omega}_n)$ in the same way that the flat differential triad $(\mathcal{M}, \Omega, d)$ arises from an inverse system of the finsheaves $\vec{S}_n = \vec{\Omega}_n(\vec{F}_n)$, the work of Zeeman (1964) provides significant clues. The key idea from (Zeeman, 1964) for our finitary considerations here is that when causality is modeled after a partial order between events in $\mathcal{M}$, its causal automorphisms constitute a group $\mathbf{G}$ isomorphic to the conformal orthochronous Lorentz group L^+. Also, $\mathbf{G}$ is, by definition, the group of homeomorphisms of $\mathcal{M}$ regarded as a causal space having for topology the causal Alexandrov (1956a,b, 1967) one (Torretti, 1981). It follows that the maps in the finsheaf $\mathbf{G}_n$, being by definition local homeomorphisms of the qauset $\vec{\Omega}_n$, respect the local (quantum) causal topology of $\vec{\Omega}_n$ which, in turn, effectively corresponds to the generating or germ relation $\vec{\rho}$. These are the finitary (and quantal) analogues of the causal automorphisms in (Zeeman, 1964), as we argued earlier. In fact, in the next section, by a heuristic implementation of the FEP, FLP, and FLRP given in section 2, we will use these finitary causal morphisms to define a finitary, quantal, and causal gauge-theoretic version of Lorentzian gravity on the gauged $\vec{\mathcal{M}}_n$ by supplying it with a nonflat $\mathbf{g}_n$-valued spin-Lorentzian connection $\mathcal{A}_n$ and its associated curvature $\mathcal{F}_n$.

For the time being we note that the expected Minkowskian classical limit $\mathbf{G}$-sheaf $(X \subset \mathcal{M}, d, \Omega, L^+)$, being flat, admits of global sections (Mallios, 1998a), a result which in physical parlance is known by the following fact: there is a global inertial coordinate patch (frame or gauge) covering the entire flat Minkowski space (Torretti, 1981). However, in a curved spacetime M, there are only local inertial frames (gauges) covering (i.e., with origin soldered at) its point events according to the CEP. These are independent "kinematical frames" (gauge possibilities) as we said in section 2 and this "kinematical independence" or "gauge freedom" motivates us here to define a nonflat connection on (i.e., to gauge) the flat $\mathbf{G}_n$-finsheaf $\vec{\mathcal{M}}_n$. Then, the resulting gauged, hence curved, finsheaf will not admit global sections (Mallios, 1998a).

We close this section by commenting on the operational significance of our $\mathbf{G}_n$-finsheaf model of quantum causality and its (global) causal symmetries. If one takes seriously the conjecture above about the convergence of the inverse system or net $\mathcal{K} := \{\vec{\mathcal{M}}_n\}$ to the classical flat Minkowskian **G**-sheaf $(X \subset \mathcal{M}, \Omega, d, L^+)$ at maximum power of resolution or infinite localization of X into its point events *à la* Sorkin (1991) and Raptis (2000a), then sound operational meaning may be given to qausets and their finitary symmetries in complete analogy to the one given to topological poset substitutes F_n of bounded regions X of continuous spacetime manifolds M in (Sorkin, 1991, 1995) and their quantal algebraic relatives $\Omega_n(F_n)$ in (Raptis and Zapatrin, 2000, in press). Since the F_ns were seen to converge to X, they were taken to be sound approximations of its point events, whereby a coarse determination of the locus of an event x in X is modeled by an open set or region about it. We simulate this semantic model for the F_ns in the case of our qausets $\vec{\Omega}_n$ as follows: we introduce a new "observable"[67] for spacetime events called "causal potential (or propensity) relative to our locally finite (coarse) observations $\mathcal{U}_n$ of them," symbolized by $\vec{\phi}_n$, so that the causal relation $x \to y$ between two events in $\vec{F}_n$ can be read as "x has higher causal potential than y" (i.e., formally: $\vec{\phi}_n(x) > \vec{\phi}_n(y)$).[68] Thus, causality may be conceived as "causal potential difference between events relative to our observations of them."[69]

This definition of $\vec{\phi}_n$ applies in case the causet $\vec{F}_n$ is the causally interpreted finitary poset substitute F_n of a bounded spacetime region X as defined in (Raptis, 2000b). If F_n derives from the locally finite open cover $\mathcal{U}_n$ of X, $\vec{\phi}_n$ in $\vec{F}_n$ may be read as follows: the causal potential $\vec{\phi}_n$ of an event x in X relative to our observations $\mathcal{U}_n$ of X[70] corresponds to the "nerve" $\mathcal{N}$ covering x relative to $\mathcal{U}_n$,

[67] To be established as a dynamical variable $\mathcal{A}_n$ in the next section where we gauge $\vec{\mathcal{M}}_n$.

[68] This formal labeling of events by $\vec{\phi}_n$ is in complete analogy to the natural number $\mathbb{N}$-labeling of events *à la* Rideout and Sorkin (2000). There the sequential growth dynamics proposed for causets was seen to be independent from their $\mathbb{N}$-labeling, thus in some sense independent of an external (background) discrete $\mathbb{N}$-valued time gauge parameter (i.e., it is "external time-covariant"). In the next section we will argue that, similarly, the reticular gauge connection $\mathcal{D}_n$, based on which the dynamical law for qausets is expected to be formulated as an equation between sheaf morphisms (Raptis, in preparation-c), is gauge $\mathcal{U}_n$-independent, thus also $\mathcal{A}_n$-covariant. See also (Raptis, in press, 2001) for more on this, but from a more categorical or topos-theoretic perspective.

[69] In a plausible "particle interpretation" of our reticular scheme, whereby a network of causet (or qauset) connections is interpreted in the manner of Dimakis *et al*. (1995) as the reticular pattern of the dynamics of particles or quanta of causality—which may be called "causons" for obvious reason—the causal connection $x \to y$ has the following rather natural physical interpretation in terms of the causal potential $\vec{\phi}_n$: "a causon descends from the event x of higher causal potential to the event y of lower causal potential." This is in literal analogy, for instance, with the motion of an electron in an electromagnetic potential gradient (or connection!) field $\mathcal{A}$, hence the natural denomination of $\vec{\phi}_n$ as "causal potential."

[70] One can equivalently call it "the causal potential of an event x in X at the limit of resolution of X corresponding to $\mathcal{U}_n$" (Cole, 1972; Raptis, 2000a). The definition of $\vec{\phi}_n$ as being relative to our coarse spacetime observations is reflected by its index which is the same as that of the locally finite open cover $\mathcal{U}_n$ of X—a finite n signifying a pragmatic limited (finite), but at the same time coarse and perturbing, power of resolution of X into its point events.

whereby $\mathcal{N}(x) := \{U \in \mathcal{U}_n \mid x \in U\}$ (Raptis and Zapatrin, 2000).[71] Then, at the level of resolution of the spacetime manifold—now regarded as a causal space—corresponding to $\vec{F}_n$, $x \rightarrow y$ (i.e., "x causes y") means operationally that $\mathcal{N}(y) \subset \mathcal{N}(x)$ (i.e., "every (rough) observation of y is a (coarse) observation of x")[72]; thence, $\vec{\phi}_n(x) > \vec{\phi}_n(y)$.

In terms of the definition of the smallest open sets in $\mathcal{U}_n$ containing x and y, $\Lambda(x)$ and $\Lambda(y)$, given in section 3 and in (Raptis, 2000b; Sorkin, 1991), that is to say, $\Lambda(x) := \bigcap\{U \in \mathcal{U}_n \mid x \in U\} \equiv \bigcap \mathcal{N}(x)$, $\mathcal{N}(y) \subset \mathcal{N}(x)$ reads $\Lambda(x) \subset \Lambda(y)$ with "$\subset$" standing for strict set-theoretic inclusion. This is precisely how the topological partial order $\rightarrow$ in F_n was defined in (Sorkin, 1991), only in our $\vec{F}_n$ it is reinterpreted causally (Raptis, 2000b).[73] It must be mentioned that such a conception of (quantum) causality as a difference in cardinality (or degree) was first conceived in a different mathematical model by Finkelstein (1969), whereas in (Breslav *et al.*, 1999), and in a model similar to ours, the collection $\mathcal{U}_n$ of open sets were assigned to teams (or organizations) of "coarse observers" of spacetime topology and it is explicitly mentioned that the relations $x \rightarrow y$ means that "*the event x has been observed more times* (by the team of observers) *than the event y*." There, however, the "$\rightarrow$" obtained from Sorkin's "equivalence algorithm" was seen to still have its original topological meaning and it was not given a directly causal significance like in our theory.[74]

From the definitio of $\vec{\phi}_n$ above, it follows that the generator of local (i.e., contiguous) causal potential differences between events in $\vec{F}_n$ corresponds to the relation of immediate causality $\stackrel{*}{\rightarrow}$ linking events, say x and y, such that $\Delta\vec{\phi}_n(x, y) := \vec{\phi}_n(x) - \vec{\phi}_n(y) = 1$. We may symbolize this "contiguous causal potential difference"—the "local germ of the quantum causal potential," by $\vec{\phi}^*_n$. If we pass to the qauset $\vec{\Omega}_n$ associated with $\vec{F}_n$, or equivalently, to the finsheaf $\vec{S}_n$ of qausets, the aforementioned generator of causal potential differences assumes a completely algebraic expression as $\vec{\rho}$. Again, we recall from section 3 that $I_p\vec{\rho}I_q \Leftrightarrow I_pI_q(\neq I_pI_q) \stackrel{\neq}{\subset} I_p \cap I_q$ generates the quantum causal Rota topology of $\vec{S}_n$ by relating primitive ideals I_p and I_q in the primitive spectrum $\vec{\mathcal{S}}(\vec{\Omega}_n)$ $(p, q \in \vec{F}_n)$ if and only if $p \rightarrow q$ and $\nexists r \in \vec{F}_n : p \rightarrow r \rightarrow q$ (i.e., iff $p \stackrel{*}{\rightarrow} q$ in $\vec{F}_n$) (Breslav *et al.*, 1999; Raptis, 2000b).

[71] In (Raptis and Zapatrin, 2000), nerves were seen to be simplicial complexes and the topological discretization of manifolds based on them is due to Alexandrov (1956a,b, 1961).

[72] See (Breslav *et al.*, 1999) for a similar operational semantics, but applied to the topological not to the causal structure of spacetime like we do here.

[73] Note that event vertices in the causet $\vec{F}_n$ that are causally unrelated (i.e., "space-like") are covered by different nerves in $\mathcal{U}_n$ of equal simplicial degree (Raptis and Zapatrin, 2000). They are the reticular versions of equal-time spacelike 3-slices of a (globally) hyperbolic spacetime manifold.

[74] That is, the quantum observable or dynamical variable in their theory is topology proper, not local causality. See (Raptis and Zapatrin, in press) for more discussion about this distinction.

Here, in the algebraic setting of qausets, the generator of quantum causality (i.e., the germ $\vec{\phi}_n^*$ of the quantum causal potential $\vec{\phi}_n$) relative to our finitary spacetime observations in $\mathcal{U}_n$, $\vec{\rho}$, has the following operational and quantal *à la* Heisenberg (because noncommutative algebraic) meaning that reads from its very algebraic definition: point events in $\vec{\Omega}_n$, which correspond to primitive ideals in $\vec{\mathcal{S}}(\vec{\Omega}_n)$,[75] have a product ideal that is strictly included in their intersection ideal, with the "directedness" (asymmetry) of their immediate quantum causal connection, say "from-*p*-to-*q*" ($p \stackrel{*}{\rightarrow} q$), being reflected in the noncommutativity of their corresponding ideals in $\vec{\Omega}_n$ (i.e., $I_p I_q \neq I_p I_q$).[76] This operational description of

[75] Recall from (Breslav *et al.*, 1999; Raptis, 2000b; Raptis and Zaptrin, 2000) and section 3 the definition of the primitive ideals in the corresponding quantum topological $\Omega_n(F_n) : I_p := \text{span}\{|q\rangle\langle r| : |q\rangle\langle r| \neq |p\rangle\langle p|\}$; where $|q\rangle\langle r| := (q \rightarrow r) \in F_n$. Parenthetically, it is rather interesting to observe in this definition of the primitive ideals (points) in the quantal topological incidence algebras $\Omega_n(F_n)$ that the elements (ket-bras) that constitute them are quantal acts of determination of what in the classical limit space will emerge as "momentum (covector) states" and serial concatenations thereof (i.e., "spacetime time-like paths"); see physcial interpretation of the Ω^is ($i \geq 1$) in (Raptis and Zapatrin, 2000). By this very definition of the I_ps in $\mathcal{S}(\Omega_n(F_n))$, we see that the operations of determination of pure quantum spacetime states (events) in Ω_n, namely, the elements of Ω^0 (the $|p\rangle\langle p|$s in the defintion of the I_ps above; Raptis and Zapatrin, 2000), are excluded from them (Raptis and Zapatrin, in press). So, operations of determination of what classically (i.e., at the non-pragmatic decoherence limit of infinite refinement of the spacetime continuum into its point events) appear as momentum states tangent to spacetime "position states" (point events) are "incompatible" or "complementary" in Bohr's sense with (i.e., they exclude) quantum acts of localization of the latter. This observation shows that some kind of quantum uncertainty is built into our Rota algebraic scheme *ab initio* thus it further justifies the physical interpretation of the limit of infinite localization of spacetime events as Bohr's correspondence principle (Raptis and Zapatrin, 2000). The quantum character of the noncommutative toplogy generated by the local (and dynamical) quantum causality $\vec{\rho}$ is analytically studied in (Raptis, in press.)

[76] This is a first indication of a fundamental noncommutativity of (acts of localization of) "points" (i.e., "coarse spacetime events") underlying quantum causal topology in a model like ours (where "points" are represented by primitive ideals in the primitive spectra $\vec{\mathcal{S}}$ of the incidence algebras $\vec{\Omega}$ involved). In a coming paper (Raptis, in press), the incidence algebras modeling qausets here, as well as their localizations, are studied in the ligth of scheme theory (Hartshorne, 1983; Shafarevich, 1994) and a noncommutative dynamical local quantum causal topology for (at least the kinematics of) Lorentzian quantum gravity is defined based on such nonabelian schematic algebra localizations in much the same way to how Noncommutative Algebraic Geometry was defined in (Van Oystaeyen and Verschoren, 1981) based on nonabelian Polynomial Identity (PI) ring localizations—it being understood that Rota algebras can be regarded as PI rings (Freddy Van Oystaeyen in private communication). It must be a fruitful project to compare the resulting "noncommutative topology for curved quantum causality" in (Raptis, in press) with the one defined and studied in (Van Oystaeyen, 2000a). The second author (IR) wishes to thank Freddy Van Oystaeyen for motivating such a study in a crucial private communication and in two research seminars; see (Van Oystaeyen, 2000b). Ultimately, the deep connection for physics is anticipated to be one between such a noncommutative conception of the local quantum causal topology of spacetime and the fundamental microlocal quantum time asymmetry expected of "*the true quantum gravity*" (Penrose, 1987). Again, such a fundamental time asymmetry in a curved finitistic quantum causal space similar to ours has already been anticipated by Finkelstein (1988). It is also entertained in (Raptis, in press).

quantum causality in $\vec{\Omega}_n$ relative to our coarse observations of events in a bounded region of spacetime—now interpreted as a causal space, follows from the operational description of causality in the causet $\vec{F}_n$ from which it derives via the causal potential "observable" $\vec{\phi}_n$ defined above.

All in all, (quantum) causality is operationally defined and interpreted as a "power relationship" between spacetime events relative to our coarse observations (or approximate operations of local determination) of them, namely, if events x and y are coarsely determined by $\mathcal{N}(x)$ and $\mathcal{N}(y)$ with respect to $\mathcal{U}_n$, and $\mathcal{N}(y) \subset \mathcal{N}(x)$, then "$x$ causes y." The attractive feature of such a definition and interpretation of causality is that, by making it relative to $\mathcal{U}_n$, we render it "frame- or observation-dependent,"[77] ultimately, relativistic.[78]

In the same way, one can give operational meaning to the finitary local (quantum) causal automorphisms of the $\vec{\Omega}_n$s in $\vec{S}_n$ mentioned above. They represent finitary operations of determination of the local symmetries of quantum causality as encoded in the finsheaf $\vec{S}_n$ and they too are organized in the group finsheaf $\mathbf{G}_n$. The operational interpretation of the elements of $\mathbf{G}_n$ as coarse reticular (and quantal) replacements of the continuous local orthochronous Lorentz Lie symmetries of the smooth gravitational spacetime of GR will become transparent in the next section when we gauge the flat Minkowskian $\mathbf{G}_n$-finsheaf $\vec{\mathcal{M}}_n$ by providing a nonflat $\mathbf{g}_n$-valued connection 1-form $\mathcal{A}_n$ on it.

5. GAUGING QUANTUM MINKOWSKI SPACE: NONFLAT CONNECTION ON $\vec{S}_n$

The reader was prepared in the previous sections for the present one where we will attempt to curve the flat and quantal $\mathbf{G}_n$-finsheaf $\vec{\mathcal{M}}_n := \mathbf{G}_n(\vec{S}_n)$ by gauging or localizing it. As it was repeatedly mentioned earlier, this procedure is tantamount

[77] One may think of the open Us in $\mathcal{U}_n$ as some sort of "rough coordinate patches" or "coarse frames" or even as "fuzzy gauges" (Mallios, 1998a) covering or measuring coarsely (i.e., approximately localizing) the point events in X.

[78] Recall that the causal potential $\vec{\phi}_n$ of events is defined relative to our coarse observations $\mathcal{U}_n$ of them, so that, as we will see in the next section, its localization (gauging) and relativization will effectively amount to establishing a local transformation theory for it that respects its dynamics (due to a finitary sort of Lorentzian quantum gravity), in the sense that this dynamics becomes independent of the level of resolution corresponding to our observations $\mathcal{U}_n$ of spacetime into its events, or equivalently, it becomes independent of the local gauges (frames) $\mathcal{U}_n$ that one lays out to chart the spacetime events and measure, albeit coarsely, physical attributes such as the gravitational field "located there" (Mallios, 1998a). This will be then the transcription of the fundamental principle of GR, which requires that the laws of physics are invariant under the diffeomorphism group of the smooth spacetime manifold Diff(M) (i.e., the principle of General Covariance), in a sheaf-theoretic model for a curved finitary quantum causal space: "the laws of physics are equations between sheaf morphisms"—the main sheaf morphism being the connection $\mathcal{D}$ (Mallios, 1998a). We will return to this principle in section 5 where we define $\mathcal{D}_n$ as a finsheaf morphism in our scheme and further discuss its quantum physical implications in section 6.

to defining a reticular nonflat spin-Lorentzian connection $\mathcal{A}_n$[79] that takes values in the orthochronous Lie algebra $\mathbf{g}_n \equiv \ell_n^+$ of the group finsheaf $\mathbf{G}_n$ adjoint to $\vec{S}_n$ consisting of the latter's local quantum causal symmetries—the finitary substitute of the continuous orthochronous Lorentz Lie group manifold L^+ which, in turn, is the structure group of (global symmetries of) the flat $\mathbf{G}_n$-finsheaf $\vec{\mathcal{M}}_n$. The resulting curved $\mathbf{G}_n$-finsheaf $\vec{\mathcal{P}}_n := (\vec{F}_n, \vec{\Omega}_n, \mathcal{L}_n, \mathcal{D}_n := d_n + \mathcal{A}_n)$ may be regarded as a finitary, causal and quantal replacement of the classical kinematical structure $\mathcal{P}$ on which GR is formulated as a gauge theory of a spin-Lorentzian connection 1-form $\mathcal{A}$.

As it was also alluded to in the introduction, our contending that the curved $\mathbf{G}_n$-finsheaf $\vec{\mathcal{P}}_n$ is a finitary, causal and quantal replacement of the classical CartanBergmann $\mathbf{G}$-bundle $\mathcal{P} = (X, \Omega, L^+, \mathcal{D} := d + \mathcal{A})$, basically rests on the idea that an inverse system of the former curved finsheaves of qausets yields the latter as a classical gravitational spacetime structure (Raptis, 2000a) at the operationally ideal limit of finest resolution or localization of X into its point events, their causal ties and the local symmetries thereof—which limit, in turn, may be interpreted as Bohr's Correspondence Principle yielding classical structures from quantum ones (Raptis and Zapatrin, 2000).

Thus, we consider a bounded region X of a curved smooth spacetime manifold M. We assume that gravity is represented by a nonflat $sl(2, \mathbb{C})$-valued connection 1-form $\mathcal{A}$ on the curved CartanBergmann $\mathbf{G}$-bundle $\mathcal{P} = (X, \Omega, L^+, \mathcal{D} = d + \mathcal{A})$. First we discuss a mathematical technicality that our finsheaf-theoretic model should meet in order to be able to define a (nonflat) connection $\mathcal{D}_n$[80] on the (flat)

[79] The reader should note the index n given to the connection $\mathcal{A}$ that is the same as the one given to the causet $\vec{F}_n$, its associated qauset $\vec{\Omega}_n$ and the latter's local quantum causal symmetries $\ell_n^{\uparrow}$. Properly viewed, the connection $\mathcal{A}$ on the $\mathbf{G}_n$-finsheaf $\vec{\mathcal{M}}_n := (\vec{F}_n, \vec{\Omega}_n, d_n, \mathcal{L}_n)$ in focus inherits the latter's "finite degree or energy of resolution n" of the region X of the curved spacetime manifold M by our coarse observations $\mathcal{U}_n$ of its events, their causal ties and the symmetries of the latter (Cole, 1972; Raptis, 2000a). The reader should notice that the index n is also given to the reticular Kähler-Cartan differential d in $\vec{\mathcal{M}}_n$ just to remind one of its discrete character *à la* Dimakis *et al.* (1995).

[80] Note that until now we used the gauge potential $\mathcal{A}$ for the mathematical concept of connection $\mathcal{D}$, when, in fact, $\mathcal{A}$ is just the part of $\mathcal{D} = d + \mathcal{A}$ that makes it nontrivial (i.e., nonflat) (Mallios, 1998a). This is the physicist's "abuse" of the concept of connection, presumably due to his rather "pragmatic" or at least "practical" attitude towards mathematics, namely, that he is interested on the part of $\mathcal{D}$ that is responsible for curvature (which can be physically interpreted as the gauge potential of a physical force). In fact, the substitution $d \to \mathcal{D} = d + \mathcal{A}$ is coined "gauging" in physics jargon, when d is from a mathematical point of view a perfectly legitimate connection; albeit, a trivial (i.e., flat) one (Mallios, 1998a). The same "abuse" of $\mathcal{D}$ is encountered in (Baez and Muniain, 1994) where only the gauge potential $\mathcal{A}$ is coined "connection." Here, we too adopt a physicist's approach and by "gauging our flat Minkowskian principal finsheaf $\vec{\mathcal{M}}_n$" essentially we mean "adjoining a nonzero connection term $\mathcal{A}_n$ to its flat differential d_n." This asymphony between the mathematician's and the physicist's conception of the notion of connection aside, one should always keep in mind that $\mathcal{D}$ is a generalized differential operator, with its nonzero part $\mathcal{A}$ generalizing or "extending" by the process of gauging the usual flat differential operator d.

finsheaf $\vec{\mathcal{M}}_n$. Two sufficient conditions for the existence of a connection $\mathcal{D}$ on an algebra or vector sheaf or bundle over a manifold M, regarded as a topological space, are that M is paracompact and Hausdorff[81] (Mallios, 1998a). It is expected that, since our finsheaves of qausets are finitary (and quantal) replacements of an at least T_1[82] and relatively compact[83] topological space X (Raptis, 2000a; Sorkin, 1991), if we relax T_2 to T_1 and paracompactness to relative compactness, we are still able to define a connection $\mathcal{D}_n$ on a vector or algebra sheaf over it such as our $\vec{\mathcal{M}}_n$ (Raptis, 2000a). So $\mathcal{D}_n$ exists (i.e., it is "defineable") on $\vec{\mathcal{M}}_n$. In fact, we know that d_n is already defined on the qauset stalks of $\vec{\mathcal{M}}_n$ *à la* Dimakis and Müller-Hoissen (1999) or Zapatrin (in press), and that it effects sub-sheaf morphisms $d_n : \vec{\Omega}^i_n \to \vec{\Omega}^{i+1}_n$ there (Mallios, 1998a); albeit, it is a flat connection (Dimakis and Müller-Hoissen, 1999; Mallios, 1998a). In turn, this $\mathcal{D}_n = d_n$ on the finitary, causal, and quantal Minkowskian finsheaf $\vec{\mathcal{M}}_n$ means that $\mathcal{A}_n = 0$ throughout $\vec{\mathcal{M}}_n$, so that by our physical terminology the latter is an ungauged, thus flat, finsheaf.[84]

To curve the flat finsheaf $\vec{\mathcal{M}}_n$ by adjoining to its flat connection d_n a nonzero term $\mathcal{A}_n$, we immitate in our finitary context how the curved smooth spacetime manifold M of GR may be thought of as the result of localizing or gauging the flat Minkowski space $\mathcal{M}$ of SR. Locally, (i.e., event-wise), one raises as a "vertical structure" an isomorphic copy of $\mathcal{M}$ over each spacetime event $x \in X \subset M$, thus implementing the CEP of section 2. Hence formally, $\mathcal{M}$ acquires an event-index $x(\forall_x \in X)$, $\mathcal{M}_x$, and may be regarded as some kind of fiber space over x. In view of the differential (i.e., C^∞-smooth) character of M, which in turn may be thought of as implementing the CLP discussed in section 2 (Einstein, 1924), $\mathcal{M}_x$ is geometrically interpreted as the space tangent to M at x. Totally, $TM := \bigcup_{x \in M} \mathcal{M}_x$ is the locally Minkowskian tangent bundle of M (Göckeler and Schücker, 1990) having for fibers $\mathcal{M}_x$—local isomorphs of flat Minkowski space.

Then, the term "gauging" effectively corresponds to regarding these local isomorphs of flat Minkowski space as "independent kinematical worlds," in the sense that two vectors v and v' living in the vector space fibers $\mathcal{M}_x$ and $\mathcal{M}_{x'}$,

[81] We recall that a topological space M is said to be paracompact if every open cover of it admits a locally finite refinement. Also, M is said to be Hausdorff or T_2 when it satisfies the second axiom of separation of point set topology which holds that every pair of points of M have nonintersecting (disjoint) open neighborhoods about them.

[82] We recall that a topological space X is said to be T_1 if for every pair of points x and y in it there exist open neighborhoods O_x and O_y containing them such that $x \notin O_y$ and $y \notin O_x$.

[83] As it was also noted earlier, a topological space X is said to be relatively compact, or bounded in the sense of (Sorkin, 1991), when its closure is compact.

[84] As we also mentioned in the previous section, the important point to retain from the discussion above is that in a sheaf-theoretic context like ours the role of connection $\mathcal{D}$ is as a sheaf morphism (Mallios, 1998a). We will come back to it shortly when we formulate a finsheaf-theoretic version of the Principle of General Covariance of GR in the gauged $\vec{\mathcal{M}}_n$ to $\vec{\mathcal{P}}_n$.

respectively, are "incomparable," in that one is not allowed to form linear combinations thereof.[85] Alternatively, one may describe this in a more geometrical way by saying that in a gauged space, such as the vector bundle *TM* (Göckeler and Schücker, 1990), there is no natural relation of distant parallelism or linear superposition between the elements residing in its fibers. A "rule" that enables one to compare vectors at different fiber spaces, thus it establishes some kind of relation of distant parallelism or, linear algebraically speaking, "distant linear combineability" in *TM*, is provided by the concept of connection $\mathcal{D}$ (Mallios, 1998a). The geometrical interpretation of $\mathcal{D}$, and one which shows an apparent dependence of this concept on the background geometric spacetime manifold M[86] is as a parallel transporter of vectors along smooth curves in M joining x with x'. Then, curvature $\mathcal{F}$, in the classical model for spacetime corresponding to the differential manifold M, is geometrically conceived as the Wilson anholonomy of $\mathcal{D}$ when the latter transports vectors parallely along smooth closed curves (C^∞-loops) in M—certainly a nonlocal conception of the action of $\mathcal{D}$.[87]

The second problem that we face is one of physical semantics: we want to interpret the nonflat part $\mathcal{A}_n$ of $\mathcal{D}_n$ in a finitary causal way. In the classical curved spacetime model $\mathcal{P}$, $\mathcal{A}$, apart from its usual interpretation as the gravitational gauge potential, may be physically interpreted as the smooth dynamically variable (field of the) local causal connections between the events of the C^∞-smooth spacetime region X. Since the fibers of the curved *TM* (or the Minkowskian covectors in the Ω^1 sub-bundle of $\mathcal{P}$) are local isomorphs of flat Minkowski space, the action of the spin-Lorentzian gravitational connection 1-form $\mathcal{A}$ on Minkowski vectors living in *TM*'s fibers, besides its geometrical interpretation as "parallel translation" above,

[85] For instance, one is not supposed to be able to compute their difference $v' - v$ which is the crucial operation for defining the differential operator d in general.

[86] And we say "apparent," because, as we will see shortly, in our scheme $\mathcal{D}_n := d_n + \mathcal{A}_n$ does not depend essentially on the geometric base spacetime, since it derives locally from the very algebraic structure of the stalks of the finsheaf of qausets (i.e., from the structure of the quantally and causally interpreted incidence algebras). This is the most important lesson to be learned from the Abstract Differential Geometry theory developed in (Mallios, 1998a,b), namely, that $\mathcal{D}$, the main object with which one can actually do Differential Geometry, is of an algebraic (i.e., analytic) nature and does not depend on any sort of "ambient geometric space." For instance, the two global (topological) conditions used in the conventional Calculus on manifolds to establish the existence of $\mathcal{D}$ on M, namely, that the latter is a paracompact and Hausdorff topological space, are sufficient, but by no means necessary. Such an independence is welcome from the point of view of both classical and quantum gravity where the spacetime manifold, regarded as an inert geometrical background base space, has shown to us its pathological, "unphysical nature" in the form of singularities and the nonrenormalizable infinities that plague the field theories defined on it (Mallios, in press, in preparation).

[87] In contradistinction to this classical geometric conception of curvature, in the sense that it depends on the existence of spacetime loops in M and that it is the effect of the action of $\mathcal{D}$ as the parallel transporter of geometric entities (smooth tensor fields) along them, we will be able to give shortly a purely local sort of curvature $\mathcal{F}_n$ stalk-wise in our gauged finsheaves of qausets.

may be alternatively interpreted in a causal way as follows: point-wise on the curve along which the vectors are transported, the transitive, inertial Minkowskian causality $\rightarrow$ is preserved.[88] Equivalently put, if x is a point in the curve and $v(x) \subset \mathcal{M}_x$ is the value of a vector field v at x,[89] the "coupling" $\mathcal{A}(x)[v(x)]$ may be thought of as a local Lorentz transformation (i.e., an infinitesimal spacetime isometry) of $v(x)$; hence, by definition, it preserves the local causal structure of the curved *TM*, namely, the Minkowski lightcone based at (or with origin) $x \in \mathcal{M}_x$. In this sense one may equivalently interpret the gravitational gauge connection $\mathcal{A}$ as the dynamical field of local causality, as we noted in section 2. We adopt this physical interpretation for our finitary $\mathcal{A}_n$.

Now we come to the crucial point of the present paper that was briefly mentioned at the end of section 2 and in footnote 88 above concerning metric connections. From a causal perspective, like the one we have adopted here, a sheaf may be the "natural" or "proper" mathematical structure to model such (dynamically variable) local causal connections such as $\mathcal{A}$, because of the following rather heuristic argument which in a sense motivated us to study finsheaves of qausets in the first place: by definition, a sheaf is a local homeomorphism (Bredon, 1967; Mac Lane and Moerdijk, 1992; Mallios, 1998a; Raptis, 2000a), so that when one is interested in the (dynamically varying) causal topology of spacetime like we are in the present paper where our $\mathbf{G}_n$-finsheaf of qausets is supposed to be the "quantum discretization" (Raptis and Zapatrin, 2000) of the local causal topology (i.e., the causal connections between events) and its local symmetries of a bounded region X of a curved smooth spacetime manifold M, a sheaf preserves precisely the generating relations or germs of the local causal topology of the base space. But, in our case, the latter are precisely the immediate causality (contiguity or covering) relations $\stackrel{*}{\rightarrow}$ in the causal set $\vec{F}_n$ that are mapped by the sheaf (regarded as a local homeomorphism $\vec{S}$) to the germ relations $\vec{\rho}$ of the quantum causal Rota topology of the qausets $\vec{\Omega}_n$, thus defining the finsheaf $\vec{S}_n$ of qausets over the corresponding causet.

Also, the adjoint sheaf $\mathcal{L}_n$, it too regarded as a local homeomorphism $\vec{\lambda}$ preserves the generator (i.e., the generating relation or "local germ") $\vec{\rho}$ of the quantum causal topology of $\vec{\Omega}_n$, thus it consists of local, finitary causal and quantal versions of the orthochronous Lorentz group $L^+ = SO(1,3)^{\uparrow}$. Altogether, a local $\vec{\Omega}^1_n$-section[90] of the $\mathbf{G}_n$-finsheaf $\mathcal{L}_n(\vec{S}_n)$ associates, via the composition $\vec{\lambda}_n \circ \vec{s}_n$ of the two local homeomorphisms defining the finitary sheaf $\vec{S}_n$ and its adjoint group sheaf $\mathcal{L}_n$, with a contiguous causal arrow $x \stackrel{*}{\rightarrow} y$ in the causet $\vec{F}_n$ a reticular

[88] One may conceive in this local-causal sense the standard requirement in GR that "the connection is compatible with the metric tensor field $g_{\mu\nu}$" (i.e., that $\mathcal{D}$ is a metric connection).

[89] Technically, a vector field is a cross-section of the vector bundle *TM* (Göckeler and Schücker, 1990; Von Westenholz, 1981).

[90] The reader should note the arrow over the sub-sheaf space Ω^1_n of discrete 1-forms in $\vec{\mathcal{M}}_n$ which again shows its causal interpretation, as well as its finite degree or energy of resolution index n.

Lorentz local (infinitesimal) transformation in $[\ell_n^+]_x$ which, in turn, may be thought of as "reticularly rotating" or "finitarily boosting" the quantal Minkowskian vectors in the stalk $[\vec{\Omega}_n]_{[x]}$[91] of $\vec{\Omega}_n(\vec{F}_n)$ over the event x. Thus we see how natural it is to define a finitary spin-Lorentzian connection $\mathcal{A}_n$ as a local $\vec{\Omega}_n^1$-section of the $\mathbf{G}_n$-finsheaf $(\vec{F}_n, \vec{\Omega}_n, d_n, \mathcal{L}_n) =: \vec{\mathcal{M}}_n$.

However, as we said earlier, since the latter is flat, it admits global sections (Mallios, 1998a). Flatness means that $\mathcal{A}_n \equiv 0$ throughout $\vec{\mathcal{M}}_n$ (Mallios, 1998a), or equivalently, that "the connection is identically equal to the trivial constant zero global $\vec{\Omega}_n^1$-section of the $\mathbf{G}_n$-finsheaf $\vec{\mathcal{M}}_n$." In our finitary causal context, we attribute this to the constancy (i.e., the nondynamical character) and the transitivity of the inertial Minkowskian causal connection $\rightarrow$ in $\vec{F}_n$, a property that is certainly nonlocal (Finkelstein, 1988; Raptis, 2000b).[92] In fact, the "unphysicality" of a crystalline rigid causality relation modeled after a transitive, and due to this, global partial order, is already implicitly noted by Zeeman (1964) and explicitly by Finkelstein (1988) who also emphasized the need for a dynamical finitary local causal topology (i.e., effectively for a nontrivial connection on a causal finsheaf, as propounded here).[93]

Indeed, like in Finkelstein (1988, 1991), we regard the germ relation $\stackrel{*}{\rightarrow}$ of the local causal topology of $\vec{F}_n$, or its finsheaf $\vec{s}$-image $\vec{\rho}$ of $\vec{\Omega}_n$, as being dynamically variable—a "quantum observable" (Raptis and Zapatrin, in press). This is achieved by localizing or gauging the qauset finsheaf $\vec{S}_n$ and its adjoint $\mathcal{L}_n$[94] which, in turn, corresponds to implementing a nonzero (nonflat) dynamically variable $\mathbf{g}_n$-valued gauge connection $\mathcal{A}_n$ realized as a local $\vec{\Omega}_n^1$-section of the $\mathbf{G}_n$-finsheaf $\vec{\mathcal{P}}_n = (\vec{F}_n, \vec{\Omega}_n, \mathcal{L}_n, \mathcal{D}_n)$. Thus, $\mathcal{A}_n$ effectively represents a finitary gravitational dynamics of qausets.

Since $\mathcal{A}_n$ represents the dynamics of the germ $\vec{\rho}$ of the quantum causal topology of the qauset stalks of $\vec{\mathcal{P}}_n$, it is defined locally,[95] thus purely algebraically.[96] The detailed algebraic argument that leads to the expression for $\mathcal{A}_n$ in terms of $\vec{\rho}$ is left for (Raptis, 2001). For the present paper it suffices to give the usual gauge-theoretic expression for the curvature associated with $\mathcal{D}_n : \mathcal{F}_n := \mathcal{D}_n^2 = \mathcal{D}_n \wedge \mathcal{D}_n = [\mathcal{D}_n, \mathcal{D}_n]$[97] (Baez and Muniain, 1994; Dimakis and Müller-Hoissen, 1999; Göckeler and Schücker, 1990; Mallios, 1998a) and note that it is defined entirely locally-algebraically stalk-wise in the sheaf without reference to any

[91] See (Raptis, 2000a) for notation and analytical definition of stalks of the finsheaves $S_n(F_n)$.

[92] See the CEP and its finitary formulation FEP in section 2.

[93] See also opening quotation from (Finkelstein, 1991).

[94] Of course, when one localizes or gauges qausets, it follows that their local quantum causal symmetries are gauged as well.

[95] That is to say, stalk-wise in the sheaf.

[96] Since the stalks are the causally and quantally interpreted incidence Rota algebras.

[97] Where "$\wedge$" denotes Cartan's exterior product and "[. , .]" "commutator." It follows that $\mathcal{F}_n$ is a ℓ_n^+-valued section of the $\vec{\Omega}_n^2$, sub-sheaf of $\vec{\mathcal{P}}_n$, as in the usual differential calculus on manifolds.

loop-anholonomy with respect to an ambient smooth geometric base spacetime. As we argued earlier, this is quite welcome from the point of view of quantum gravity. $\mathcal{F}_n$ may be physically interpreted rather freely in our scheme as a finitary, causal and quantal expression of Lorentzian gravity.

Now that we have mathematically defined and physically interpreted $\mathcal{A}_n$ (and its curvature $\mathcal{F}_n$) on $\vec{\mathcal{P}}_n$, we give an alternative physical interpretation for it more in line with the operational interpretation of finsheaves in (Raptis, 2000a), whereby, the latter were regarded as "approximations of the continuous spacetime observables." So again, let X be a bounded region in a curved smooth spacetime manifold M on which $\mathcal{A}$ lives (in $\mathcal{P}$). As in (Sorkin, 1991) the open sets U in the locally finite open cover $\mathcal{U}_n$ of X were physically interpreted as "coarse acts of localization (local determination) of the continuous topology carried by X's point events" which, in the finsheaf $S_n(F_n)$ of (Raptis, 2000a), translates to "coarse acts of local determination of the continuous (i.e., C^0-topological) spacetime observables,"[98] so similarly we lay out $\mathcal{U}_n$ to "measure" or chart roughly the causal topology and the causal symmetries of the bounded spacetime region X in the gravitational spacetime M.[99] Then, we organize our observations of the dynamics of local quantum causality into the curved $\mathbf{G}_n$-finsheaf of qausets $\vec{\mathcal{P}}_n$ as described above. In $\vec{\mathcal{P}}_n$, we perceive as "gravitational gauge potentials $\mathcal{A}$" the $\mathbf{g}_n$-valued $\vec{\Omega}^1_n$-sections $\mathcal{A}_n$.

So, in the manner that we described the *aufbau* of $\vec{\mathcal{P}}_n$ in sections 2–5, it is straightforward to interpret $\mathcal{A}_n$ as "equivalence classes of gravitational gauge potentials" relative to our coarse and dynamically perturbing observations $\mathcal{U}_n$ of the spacetime's causal topology. Equivalently, following verbatim the physical interpretation of finsheaves in (Raptis, 2000a), $\mathcal{A}_n([x])$ stands for a collection of gravitational gauge potentials that are "indistinguishable" at the finite level n of resolution of X into its point events.[100] "Indistinguishability" may be physically interpreted here in a dynamical way as follows: the gravitational field is not perceived as varying between any two events in the same equivalence class $[x]$.[101] Furthermore, and here is the operational weight that the sheaf-theoretic scheme of ours carries, it is our coarse operations of determination of the dynamical local quantum causal topology, which are organized into $\vec{\mathcal{P}}_n$, that are effectively encoded in $\mathcal{A}_n$, so that, by the end of the day, it is not the point events of X *per se*

[98] Which by definition preserve the local Euclidean manifold topology of X.

[99] Following the terminology in Mallios (1998a), we call the elements U of $\mathcal{U}_n$ "coarse (or fuzzy) local gauges," since they stand for rough acts of measurement of the local structure of X.

[100] Thus, we tacitly assume that the spacetime events are not only surrogate carriers of X's physical topology (Raptis, 2000a; Sorkin, 1991), but also of its other physically observable attributes—the gravitational or "locality" field being the one in focus here.

[101] This interpretation is consistent with our FEP of section 2 which, in effect, held that in a finsheaf over a causet $\vec{F}_n$ obtained by $\mathcal{U}_n$ as briefly described in sections 3 and 4, all the frames in the "fuzzy" or "coarse stalks" $\vec{\Omega}_n([x])$ over $[x]$ are inertial (because "indistinguishable by gravity") relative to each other, so to speak.

that carry information about the dynamics of (quantum) causality; rather, it is our own dynamically perturbing observations of them that create "it."[102] Then, the general relativistic character of our sheaf-theoretic scheme may be summarized in the following: the finitary gravitational connection $\mathcal{D}_n$ on $\vec{\mathcal{P}}_n$ is a sheaf morphism (Mallios, 1998a), which means that the dynamics of local quantum causality is $\mathcal{U}_n$-independent after all.[103] This is the (fin)sheaf-theoretic version of the principle of General Covariance of GR with a strong local-quantal flavor.[104]

6. DISCUSSION OF THE SOUNDNESS OF $\vec{\mathcal{P}}_n$ AND OTHER RELEVANT ISSUES

In this last section of the paper we present four arguments that support that $\vec{\mathcal{P}}_n$ is a sound model of a finitary, causal and quantal version of Lorentzian gravity.

(a) The FEP of section 2 is satisfied in $\vec{\mathcal{P}}_n$, since the latter's stalks $\vec{\Omega}_n$ are local isomorphs of a locally finite, causal and quantal version of Minkowski space (sections 3 and 4).

The FLRP of section 2 holds in $\vec{\mathcal{P}}_n$, since the latter's group $\mathbf{G}_n$-stalks are finitary, causal, and quantal versions of the orthochronous Lorentz structure group L^+ of local causal symmetries of GR.

The FLP of section 2 is satisfied in $\vec{\mathcal{P}}_n$, since the latter's qauset stalks are sound models of local quantum causality (section 3 and (Raptis, 2000b)).

The FLSP of section 2 holds in $\vec{\mathcal{P}}_n$, since the qausets residing in its stalks coherently superpose with each other (sections 3, 4, and (Raptis, 2000b)).

In section 2 we posited that a sound mathematical model of (the kinematics of) a finitary curved quantum causal space should meet structurally these four "physical axioms." Indeed, the structure of $\vec{\mathcal{P}}_n$ does meet them.

(b) $\vec{\mathcal{P}}_n$ is an algebraic model for finitary, local, and dynamically variable quantum causality which inherits its operational meaning from the pragmatic

[102] See the active operational interpretation of dynamical local quantum causality at the end of section 4. As it was shown in (Raptis, 2000a), it is not the point events *per se* that are the carriers of the continuum's topology as assumed in (Sorkin, 1991), but the (sheaves of algebras of) continuous observables that occupy this apparently existing continuum. There is no spacetime *per se*; rather, it is from the dynamical relations between (i.e., the algebraic structure of) the objects that inhabit "it" that "its" properties are extracted (Mallios, 1998a, in press, in preparation; Raptis, in press).

[103] For a short discussion of this apparently paradoxical situation, namely, that our own coarse local observations $\mathcal{U}_n$ create the dynamical local quantum causality whose dynamics is subsequently expressed in a $\mathcal{U}_n$-invariant (i.e., gauge independentor "covariant") way, see (c) in the next section.

[104] Since $\mathcal{D}_n$ respects the linear quantum kinematical structure (i.e., the coherent quantum superpositions of quasets) stalk-wise in $\vec{\mathcal{P}}_n$.

and quantal interpretation given to quantum topological incidence algebras in (Raptis and Zapatrin, 2000), hence also to their causal relatives in (Raptis, 2000b).[105] Moreover, from (Raptis and Zapatrin, 2000) it inherits its essentially alocal character, whereas, together with the physical interpretation of finsheaves (of algebras) in (Raptis, 2000a), it manifests its essential noncommitment to spacetime as an ether-like background geometrical C^{∞}-smooth base point set manifold.

(c) The local structure of classical gravitational spacetime, namely, the event and the space of Minkowskian directions tangent to it, arise only at the operationally ideal limit of infinite localization[106] of an inverse system of $\vec{\mathcal{P}}_n$s (Raptis, 2000a). The latter limit, yielding the classical gravitational sheaf or bundle $\mathcal{P}$ in a manner analogous to how the sheaf $S(X)$ of continuous functions on a topological spacetime manifold arises at infinite refinement of topological finsheaves S_n similar to our causal $\vec{\mathcal{P}}_n$s (Raptis, 2000a), may be physically interpreted as Bohr's Correspondence Principle (Raptis and Zapatrin, 2000). This further supports the quantal character of $\vec{\mathcal{P}}_n$.

All in all, putting together the physical interpretations of the theoretical schemes proposed in (Raptis and Zapatrin, 2000, in press), (Raptis, 2000b), and (Raptis, 2000a) that are amalgamated into our model $\vec{\mathcal{P}}_n$ for the dynamics of finitary quantum causality as described in sections 3–5, we may summarize the physical interpretation of $\vec{\mathcal{P}}_n$ to the following: it represents alocal, discrete, causal, and quantal operations of determination of the dynamics of causality and its symmetries in a bounded region of a curved smooth spacetime manifold with the latter not existing in a physically significant sense,[107] but only viewed as providing a surrogate scaffolding on which we base (i.e., locally solder our own operations of observing "it," which are then suitably organized into algebra finsheaves. This collective physical interpretation of $\vec{\mathcal{P}}_n$ is well in accord with the general philosophy of QT holding that inert, background, geometrical "state spaces" and their structures, such as spacetime and its causal structure, "dissolve away," so that what remains and is of physical significance, the "physically real" so to speak, is (the algebraic mechanism of) our own actions of observing "it" (Finkelstein, 1996).

[105] See also (Raptis and Zapatrin, in press).

[106] That is, at the operationally ideal situation of employment of an infinite power (or energy) to resolve spacetime into its point events (Cole, 1972). As we explained in section 3, this is theoretically implausible too due to the fundamental conflict of the principles of Equivalence and Uncertainty on which gravity and the quantum are founded at energies (i.e., microscopic powers of resolution) higher than $E_P = \hbar t_P^{-1} \approx 10^{19} GeV$—the natural cut-off of quantum gravity.

[107] Not being "physically real," so to speak.

In section 4 we stretched even further this "observer dependent physical reality" essence of quantum mechanics to an "observation created physical causality" with the introduction of the "quantum causal potential relative to our coarse observations" observable which was subsequently seen to be the dynamically variable entity represented by the finitary connection $\mathcal{A}_n$ on $\vec{\mathcal{P}}_n$ only to find that a (fin)sheaf-theoretic version of the principle of General Covariance of GR holds in our model, namely, that dynamics is categorically formulated in terms of equations between sheaf morphisms thus involving the connection $\mathcal{D}_n$ which is the main finsheaf morphism in $\vec{\mathcal{P}}_n$ (Mallios, 1998a). Hence, our mathematical expressions of "physical laws" are not observation dependent.[108] This points to the following seemingly paradoxical interpretation of our scheme: the observer acts as a "law-maker" when she observes[109] and as a "law-seeker" when she communicates her observations.[110] There is no conflict, as Finkelstein convincingly argues in (1996). After all, such an apparently conflicting "duality" may ultimately prove to be necessary for a genuine synthesis of the quantum with relativity (Finkelstein, 1996)—a synthesis which appears to be at the heart of the problem of quantum gravity *per se.*

(d) Causet theory (Bombelli *et al.*, 1987; Sorkin, 1990a,b) addresses the problem of "quantum gravity" in locally finite, causal, and to some extent, quantal terms[111] from a nonoperational pseudo-realistic point of view (Sorkin, 1995). On the other hand, Finkelstein's Quantum Net Dynamics (1988, 1989, 1991) and its subsequent generalization, Quantum Relativity Theory (1996), address the same problem in almost the same terms, but from an "entirely operational"[112] point of view. Our finsheaf-theoretic model for finitary and causal Lorentzian quantum gravity brings together Finkelstein's and Sorkin's approaches under a "purely algebraic roof" and to some extent vindicates their fundamental insight that the problem of quantum gravity may be solved or, at least, be better understood, if it is formulated as the dynamics of an atomistic local quantum causal topology.

[108] That is to say, physical laws are $\mathcal{U}_n$-gauge independent of or "invariant under" (i.e., "covariant with") our coarse and dynamically perturbing measurements of the local observables of "spacetime" (Mallios, 1998a, in press, in preparation).

[109] By establishing causal connections between the events that she observes.

[110] That is to say, when she "objectifies" her actions of determination of "it" to other observers by organizing the coarse causal nexus she has perceived in "it" to structures (sheaves) so that the dynamics $\mathcal{D}_n$ of this local causal nexus (i.e., the dynamical local causal topology) is independent of her "subjective" coarse measurements U in $\mathcal{U}_n$ of "it all."

[111] For instance, a quantum dynamics for causets is sought after a covariant path integral or "sum over causet histories" scenario (Sorkin, 1990a,b).

[112] In fact, "pragmatic."

At least, it certainly goes some way towards vindicating Einstein's hunch, that: *"Perhaps the success of the Heisenberg method points to a purely algebraic method of description of nature, that is to the elimination of continuous functions from physics"* (Einstein, 1936), and it accords with his more general and imperative intuition later on (Einstein, 1956), that:

> . . . One can give good reasons why reality cannot at all be represented by a continuous field. From the quantum phenomena it appears to follow with certainty that a finite system of finite energy can be completely described by a finite set of numbers (quantum numbers). This does not seem to be in accordance with a continuum theory, and must lead to an attempt to find a purely algebraic theory for the description of reality. But nobody knows how to obtain the basis of such a theory.

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REFERENCES

Alexandrov, A. D. (1956a). *Helvetica Physica Acta* (Suppl.) **4**, 44.
Alexandrov, P. S. (1956b). *Combinatorial Topology*, Vol. 1, Greylock, Rochester, New York.
Alexandrov, P. S. (1961). *Elementary Concepts of Topology*, Dover, New York.
Alexandrov, A. D. (1967). *Canadian Journal of Mathematics* **19**, 1119.
Ashtekar, A. (1986). *Physical Review Letters* **57**, 2244.
Baez, J. C. and Muniain, J. P. (1994). *Gauge Fields, Knots and Quantum Gravity*, World Scientific, Singapore.
Bergmann, P. G. (1957). *Physical Review* **107**, 624.
Bombelli, L., Lee, J., Meyer, D., and Sorkin, R. D. (1987). *Physical Review Letters* **59**, 521.
Bombelli, L. and Meyer, D. (1989). *Physics Letters A* **141**, 226.
Bredon, G. E. (1967). *Sheaf Theory*, McGraw-Hill, New York.
Breslav, R. B., Parfionov, G. N., and Zapatrin, R. R. (1999). *Hadronic Journal* **22**, 225.
Butterfield, J. and Isham, C. J. (2000). *Some Possible Roles for Topos Theory in Quantum Theory and Quantum Gravity*, Imperial/TP/98-99/76, to appear in a special *Festschrift* issue of *Foundations of Physics* in honour of Marisa Dalla Chiara; e-print: gr-qc/9910005.
Candelas, P. and Sciama, D. W. (1983). *Physical Review D* **27**, 1715.
Cole, E. A. B. (1972). *International Journal of Theoretical Physics* **5**, 3.
Dimakis, A. and Müller-Hoissen, F. (1999). *Journal of Mathematical Physics* **40**, 1518.

Dimakis, A., Müller-Hoissen, F., and Vanderseypen, F. (1995). *Journal of Mathematical Physics* **36**, 3771.

Donoghue, J. F., Holstein, B. R., and Robinett, R. W. (1984). *Physical Review D* **30**, 2561.

Donoghue, J. F., Holstein, B. R., and Robinett, R. W. (1985). *General Relativity and Gravitation* **17**, 207.

Einstein, A. (1924). *Schweizerische Naturforschende Gesellschaft Verhanflungen, Vol. 105*, pp. 85–93 ["*On the Ether*" in *The Philosophy of Vacuum*, S. Saunders and H. Brown, eds., English translation by S. Saunders, Clarendon Press, Oxford, 1991].

Einstein, A. (1936). *Journal of the Franklin Institute* **221**, 313.

Einstein, A. (1956). *The Meaning of Relativity*, 5th edn., Princeton University Press, Princeton.

Finkelstein, D. (1969). *Physical Review* **184**, 1261.

Finkelstein, D. (1988). *International Journal of Theoretical Physics* **27**, 473.

Finkelstein, D. (1989). *International Journal of Theoretical Physics* **28**, 441.

Finkelstein, D. (1991). *Theory of Vacuum* in *The Philosophy of Vacuum*, S. Saunders and H. Brown, eds., Clarendon Press, Oxford.

Finkelstein, D. (1996). *Quantum Relativity: A Synthesis of the Ideas of Einstein and Heisenberg*, Springer-Verlag, Berlin–Heidelberg–New York.

Göckeler, M. and Schücker, T. (1990). *Differential Geometry, Gauge Theories and Gravity*, Cambridge University Press, Cambridge.

Hartshorne, R. (1983). *Algebraic Geometry*, 3rd edn., Springer-Verlag, New York–Heidelberg–Berlin.

Mac Lane, S. and Moerdijk, I. (1992). *Sheaves in Geometry and Logic: A First Introduction to Topos Theory*, Springer-Verlag, New York.

Mallios, A. (1998a). *Geometry of Vector Sheaves: An Axiomatic Approach to Differential Geometry*, Vols. 1 and 2, Kluwer Academic Publishers, Dordrecht (volume 3, which includes further physical applications to Yang-Mills theories and gravity, is currently in the process of publication).

Mallios, A. (1998b). *International Plaza Mathematica Japonica* **48**, 93.

Mallios, A. (in press). *Abstract Differential Geometry, General Relativity and Singularities*, invited paper in *Unresolved Problems in Mathematics for the 21st Century: A tribute to Kiyoshi Iseki's 80th birthday*, IOS Press, Amsterdam.

Mallios, A. (in preparation). *Gauge Theories from the Point of View of Abstract Differential Geometry*, third volume continuation of the two-volume book (Mallios, 1998a).

Penrose, R. (1987). *Newton, Quantum Theory and Reality*, in *300 Years of Gravitation*, S. W. Hawking and W. Israel, eds., Cambridge University Press, Cambridge.

Raptis, I. (1998). *Axiomatic Quantum Timespace Structure: A Preamble to the Quantum Topos Conception of the Vacuum*, PhD Thesis, University of Newcastle upon Tyne, UK.

Raptis, I. (2000a). *International Journal of Theoretical Physics* **39**, 1703.

Raptis, I. (2000b). *International Journal of Theoretical Physics* **39**, 1233.

Raptis, I. (in preparation-a). *Non-Classical Linear Xenomorph as Quantum Causal Space and the Quantum Topos of Finitary Spacetime Schemes of Quantum Causal Sets.*[113]

Raptis, I. (in preparation-b). *Quantum Spacetime as a Quantum Causal Set: What Happened to Operationalism?*

Raptis, I. (in press). Non-commutative topology for curved quantum causality. *International Journal of Theoretical Physics*; e-print: gr-qc/0101082.

Raptis, I. (in preparation-c). *Locally Finite, Causal and Quantal Einstein Gravity.*

Raptis, I. (2001). Reflections on a possible 'quantum topos' structure where curved quantum causality meets 'warped' quantum logic. Paper presented at the 5th biannual International Quantum

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Structures Association Conference, Cesena, Italy, March–April, 2001; to appear in a special proceedings issue of the *International Journal of Theoretical Physics.*
Raptis, I. and Zapatrin, R. R. (2000). *International Journal of Theoretical Physics* **39**, 1.
Raptis, I. and Zapatrin, R. R. (in press). Algebraic description of spacetime foam. *Classical & Quantum Gravity*; e-print: gr-qc/0102048.
Rideout, D. P. and Sorkin, R. D. (2000). *Physical Review D*, **61**, 024002; e-print: gr-qc/9904062.
Robb, A. A. (1914). *Geometry of Space and Time*, 1st edn., Cambridge University Press, Cambridge (2nd edn., CUP, 1936).
Rota, G.-C. (1968). *Zeitschrift für Wahrscheinlichkeitstheorie* **2**, 340.
Selesnick, S. A. (1991). *International Journal of Theoretical Physics* **30**, 1273.
Selesnick, S. A. (1994). *Journal of Mathematical Physics* **35**, 3936.
Selesnick, S. A. (1998). *Quanta, Logic and Spacetime: Variations on Finkelstein's Quantum Relativity*, World Scientific, Singapore.
Shafarevich, I. R. (1994). *Basic Algebraic Geometry 2*, 2nd edn., Springer-Verlag, Berlin–Heidelberg–New York.
Sorkin, R. D. (1990a). Does a discrete order underlie spacetime and its metric. In *Proceedings of the Third Canadian Conference on General Relativity and Relativistic Astrophysics*, F. Cooperstock and B. Tupper, eds., World Scientific, Singapore.
Sorkin, R. D. (1990b). Spacetime and causal sets. In *Proceedings of the SILARG VII Conference*, Cocoyoc, Mexico (pre-print).
Sorkin, R. D. (1991). *International Journal of Theoretical Physics* **30**, 923.
Sorkin, R. D. (1995). A specimen of theory construction from quantum gravity. In *The Creation of Ideas in Physics*, J. Leplin, ed., Kluwer Academic, Dordrecht.
Stanley, R. P. (1986). *Enumerative Combinatorics*, Wadsworth and Brook, Monterey, California.
Torretti, R. (1981). *Relativity and Geometry*, Dover, New York.
Van Oystaeyen, F. (2000a). *Algebraic Geometry for Associative Algebras*, Marcel Dekker, New York.
Van Oystaeyen, F. (2000b). Is the topology of nature noncommutative? *A Grothendieck-type of Scheme Theory for Schematic Algebras.* Research seminars, Mathematics Department, University of Pretoria, Feb. 2–3, 2000.
Van Oystaeyen, F. and Verschoren, A. (1981). *Non-Commutative Algebraic Geometry*, Lecture Notes in Mathematics, Vol. 887, Springer-Verlag, Berlin.
Von Westenholz, C. (1981). *Differential Forms in Mathematical Physics*, Part 5, North Holland Publishing Company, Amsterdam, New York, Oxford.
Wheeler, J. A. (1964). Geometrodynamics. In *Relativity, Groups and Topology*, C. De Witt and B. S. De Witt, eds., Gordon and Breach, London.
Zapatrin, R. R. (1998). *International Journal of Theoretical Physics* **37**, 799.
Zapatrin, R. R. (in press). Incidence algebras of simplicial complexes. *Pure Mathematics and its Applications*, to appear; e-print: math.CO/0001065.
Zeeman, E. C. (1964). *Journal of Mathematical Physics* **5**, 490.
Zeeman, E. C. (1967). *Topology* **6**, 161.